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IMAGE ANALYSIS

of

AREA-SCAN TELEVISION

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1. SUMMARY

In the study described in this report we are concerned with the problem of reproducing a two-dimensional image at a remote location by using a novel method which we shall refer to as "area-scan television". This method is described in Section 2. Very briefly, it uses a scanning aperture pattern — at both the transmitter and the receiver — which consists of many individual randomly-placed apertures which are illuminated simultaneously. These aperture patterns are referred to in Sections 2 through 6 as transmissivity patterns.

The scope of the study is delineated in Section 3. In particular, the study is restricted to black-and-white television. This restriction was made to enable us to keep the costs of the study within the limits imposed by the available funds; and not because area-scan techniques cannot be used to provide multi-colored replicas. In fact, we believe this method to be, as mentioned in Sections 8 and 9, particularly well-suited for providing such replicas.

The criteria for judging (a) the quality of a replica, (b) the practical realizability of the mechanically scanning aperture patterns employed in area-scan television, and (c) the system performance of this type of television, are described in Section 4.

We derive in Section 5 sufficient conditions for obtaining, in principle, a "perfect"[1] replica. We show that these conditions cannot be satisfied by physically realizable scanning-aperture patterns. We subsequently derive four general forms of aperture patterns that provide, in principle, perfect replicas by satisfying less restrictive sufficient conditions for perfect replication. Single-spot line-scan television is shown to be a special case of two of these general forms. The other two general forms provide two possible mathematical models of aperture patterns for area-scan television.

The relative system performance of area-scan and single-spot line-scan television — which we shall henceforth refer to simply as spot-scan

[1] This term is defined in Subsection 4.1

television — is discussed in Section 7. The criterion chosen for judging this performance is the replica signal-to-noise ratio as a function of the system parameters listed in Subsection 4. 3.

The merits of spot-scan television using mechanically scanning apertures are discussed in Section 8; and the relative merits of area-scan television and spot-scan television are discussed in Section 9. Very briefly the principal conclusions of these discussions are:

(1) The instrumentation of a spot-scan system using a pair of mechanically scanning apertures is simple and reliable. In particular it requires no high-voltage supply and no sweep circuits.

(2) The signals emitted by an area-scan television transmitter are secure; that is, a replica of the transmitted picture cannot be reproduced by intercepting these signals if one does not know exactly the form of the scanning aperture pattern used.

(3) Area-scan television can be used, when the intensity of the image to be transmitted is low, to improve the quality of the received image by reducing the effect of the noise generated in the low-level circuits of the transmitter. This improvement can be achieved at the expense of an increase in video bandwidth.

Sections 2 through 4, and 8 through 9 are essentially self-contained and can be read without referring to Sections 5 through 7.

2. DESCRIPTION OF AREA-SCAN TELEVISION

In the study described in this report, we are concerned — as already mentioned — with the problem of reproducing a two-dimensional image at a remote location by using a novel method which we shall refer to as "area-scan television". This method is described below.

The two-dimensional intensity $I(x,y,t)$ of the image (see Figure 1) is converted, effectively, to a point intensity $I_P(t)$ by performing, for example optically, the transformation

$$I_P(t') = \int_{-a'}^{a'} \int_{-b'}^{b'} I(x',y',t') W_I(x',y',t') dx' \, dy', \tag{2-1}$$

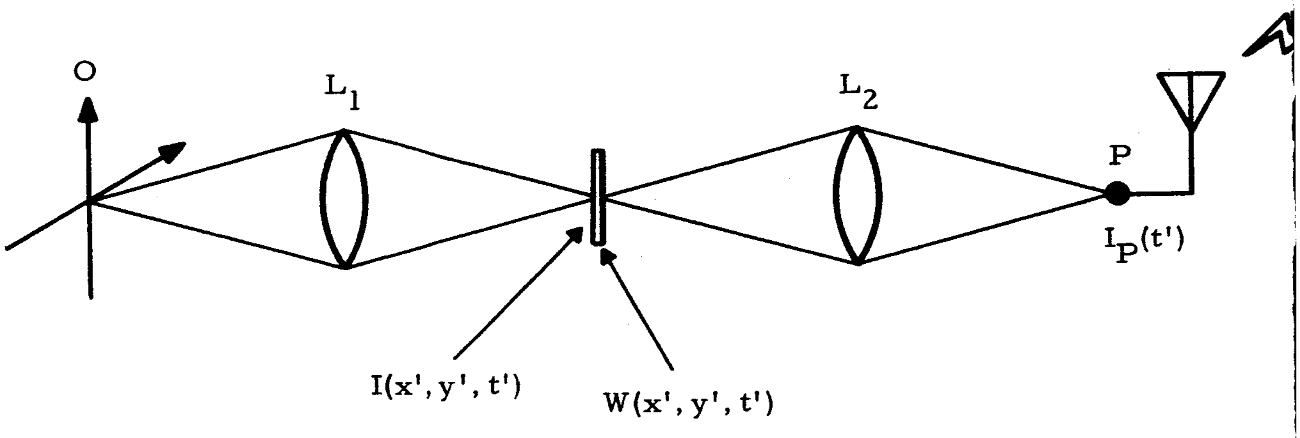
where $W_I(x',y',t')$ is a real dimensionless transmissivity function used to assign the "weight" $W_I(x',y',t')$ to the point (x',y') of the image at the instant t', and where the limits of integration determine the spatial boundaries of the image to be reproduced. (This is assumed to be rectangular.) The intensity $I_P(t')$ is in turn converted to an electrical signal $S_T(t')$ by a transducer performing the operation

$$S_T(t') = C_T I_P(t'), \tag{2-2}$$

where C_T is a dimensional constant. If we express $I(x',y',t')$ in watts/cm^2 and a',b' in centimeters, I_P must be expressed (because of equation (2-1) in watts; and hence, if we express $S(t')$ in volts, C_T has the dimensions of volts/watt. The signal $S_T(t')$ is transmitted to a remote location where the received signal, $S_R(t)$ is converted first, by a transducer, into a point intensity and then optically into a uniform two-dimensional intensity distribution $R_U(t)$ by performing the operation

$$R_U(t) = C_R S_R(t), \tag{2-3}$$

where the constant C_R must have the dimensions (watts/cm^2)/volt if we express $S_R(t)$ in volts and $R_U(t)$ in watts/cm^2. The uniform intensity distribution is converted to a succession of, in general, non-uniform intensity distributions $R(x,y,t_k)$ at instants t_k by performing the transformation



The lens L_1 forms the "original image" intensity d

The weighting function $W_I(x',y',t')$ lies in the same p

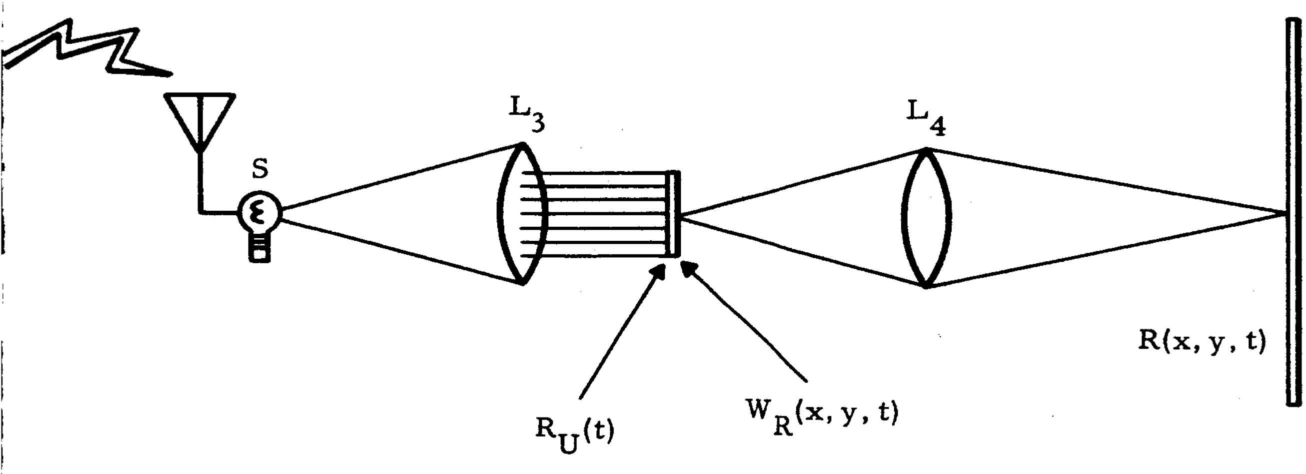
The lens L_2 converts the two-dimensional intensit to a point intensity $I_P(t')$.

The transducer — probably a photosensor P — cor is amplified and radiated by the transmitting anter amplification — to modulate the light source S.

The condenser lens L_3 produces the uniform illum

The lens L_4 is used to image the uniform illuminat $W_R(x,y,t)$, onto an integrating screen used to display

> Note: Scaling factors arising from the variou the intermediate images, and the repli analysis.



istribution $I(x', y', t')$.

lane as the original image.

y distribution $I(x', y', t')$, as seen through $W_I(x', y', t')$,

verts this intensity to an electrical signal which
na. The received signal is used — after

ination $R_U(t)$.

ion $R_U(t)$, as seen through the weighting function
the desired replica $R(x,y,t)$ of $I(x', y', t')$.

s different dimensions of the original image,
ca, are omitted in Section 2 and in the subsequent

Figure 1. Area - Scan Television Principle

$$R(x,y,t_k) = \int_{t_{k-1}}^{t_k} R_U(t)\,W_R(x,y,t)\,dt, \tag{2-4}$$

where $W_R(x,y,t)$ is a real dimensionless transmissivity function used to assign the "weight" $W_R(x,y,t)$ to the uniform intensity distribution $R_U(t)$ at the point (x,y) at the instant t. In (2-4) the multiplication may, for example, be carried out optically in which case the integration may be performed by a photo-luminescent (phosphor) screen. We choose the interval of integration

$$T \equiv t_k - t_{k-1} \tag{2-5}$$

small enough for the intensity $I(x,y,t)$ to change only insignificantly during the corresponding time interval, $(t_k - \tau) - (t_{k-1} - \tau)$ where

$$\tau \equiv t - t' \tag{2-6}$$

is the propagation time between the transmitter and the receiver[1]. We may therefore write, to a high degree of approximation,

$$I(x',y',t') = I(x',y',t_{k-1} - \tau) \qquad \text{for} \qquad t_{k-1} \leq t < t_k. \tag{2-7}$$

We assume that the successive intensity distributions $R(x,y,t_k)$ obtained for successive values of k are replicas of the corresponding intensity distributions $I(x',y',t_{k-1} - \tau)$, which are time-samples of the time-varying intensity distribution $I(x',y',t-\tau)$ of the original image. The distributions $R(x,y,t_k)$ provide a display of the original image at a remote location from this image if the above assumption is correct. One of our major objectives in this study is to discover whether area-scan television can be used to provide such replicas.

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[1] The identify sign is used in the report to denote that an equation is true by definition.

3 SCOPE OF THE STUDY

Very briefly, the general questions which we answer in the present study are:

(a) Do there exist physically realizable weighting functions $W_I(x',y',t')$ and $W_R(x,y,t)$ for which the reproduction of the original image can be made to be, in principle, a perfect replica of this image.

(b) If such weighting functions exist, how fine a resolution can one achieve with a replica obtained by using specific forms of weighting functions.

(c) What are the merits of spot-scan television using mechanically scanning apertures.

(d) What are the relative merits of area-scan television and spot-scan television.

In the study, a number of restrictive assumptions are made and the questions above are answered only for systems satisfying them.

(A) Only a "monochromatic"[1] replica of the original image is required.

(B) The replica and the uniform intensity distribution $R_U(t)$ have the same dimensions as the original image.

(C) The weighting functions $W_I(x', y', t')$ and $W_R(x,y,t)$ are instrumented by translating synchronized time-variant transmissivity patterns (aperture patterns) at a uniform velocity.

(D) These patterns are assumed to be periodic in one direction and to have a common period.

(E) The transmissivity patterns of the two weighting functions are identical.

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[1] Monochromatic in the sense of black and white conventional television; and not in the sense of using light of only a single wavelength.

The first assumption is made to enable us to keep the costs of the study within the limits imposed by the available funds; and not because the area-scan method cannot be used to provide multi-colored replicas. In fact we believe that this method is particularly well-suited for providing such replicas. The first assumption has already been implied in the description of area-scan television given earlier since no wavelength dependence is shown in the operations used to form the replica. It is also implied in the definition of a perfect replica given at the beginning of Subsection 4.1 below, because this definition does not require color matching.

The second assumption is a trivial one and is only made to simplify the exposition.

The third and fourth assumptions are made in order to restrict the analysis to the type of instrumentation which we believe to be the simplest.

The fifth assumption is made because we believe that no advantages can be obtained by using two different types of weighting functions. We do not, however, possess a rigorous proof of this statement.

4. CRITERIA

4.1 Criteria for Judging the Quality of a Replica

We say that a replica of the original image is perfect if its intensity distribution at time t_k is identical to that of the original image during the time interval

$$t_{k-1} - \tau \leq t < t_k - \tau. \tag{4-1}$$

This condition is satisfied if the "corresponding" two-dimensional spatial line spectra of these distributions are identical. The term "corresponding" is used here to indicate that these line spectra are the coefficients of two-dimensional Fourier series that have the same pair of fundamental periods.

We consider in the present report only physically realizable — and hence bandlimited — images Hence the spatial line spectra of the images considered, and their replicas, contain only a finite number of non-zero spectral lines. Nevertheless, we shall express the various functions with which we shall be concerned by Fourier series containing an infinite number of terms; it should be understood, however, that only terms corresponding to non-zero spectral lines are significant.

In practice replicas are never perfect, and we therefore need criteria for judging their quality; that is, the fidelity with which they represent the original image.

These criteria are:

(I) Resolution

(II) Contrast

(III) Relative total intensity range

(IV) Linearity with respect to intensity

(V) Spatial distortion.

Resolution will be defined later in terms of the "quality function" discussed in Subsection 6.1. This function is equivalent to the spread function used in assessing the performance of conventional optical image-reproducing systems.

4.2 Criteria for Judging Whether Weighting Functions are Practically Realizable

The criteria used to determine whether weighting functions are practically realizable are:

(1) The complexity and, in particular, the "fineness" of the transmissivity patterns used for the weighting functions.

(2) The overall length of these patterns.

(3) The speed with which these patterns must be moved.

4.3 Criteria for Judging the Relative System Performance of Area-Scan Television and Spot-Scan Television

The criterion chosen for judging the system performance of spot-scan television is the replica signal-to-noise ratio as a function of:

(1) video bandwidth

(2) integration interval

(3) noise originating in the low-level circuits of the transmitter[1]

(4) noise originating in the communication link and in the low-level circuits of the receiver

(5) transmitter power

(6) the losses in the communication link arising from the inverse square law, attenuation, and reflections.

[1]The noise originating in the low-level circuits of the transmitter consists — assuming a photosensor is used to detect the radiation from the image — of photosensor noise and of noise in the subsequent preamplifier.

5. GENERAL FORMS OF WEIGHTING FUNCTIONS PROVIDING PERFECT REPLICAS

5.1 Sufficient Conditions on the Two-Dimensional Spatial Line Spectra of the Weighting Functions for Perfect Replication

We shall derive these conditions by requiring that the "corresponding" spatial two-dimensional line spectrum of the replica be identical with that of the original image. The conditions can be derived by representing the functions $I(x',y',t')$, $R(x,y,t)$, $W_I(x',y',t')$, and $W_R(x,y,t)$, by four two-dimensional Fourier series having the same pair of fundamental periods 2a and 2b along the x and y-axes, respectively. We have[1]

$$I(x',y',t') = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t') e^{i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} \tag{5-1}$$

$$R(x,y,t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \mathcal{R}_{rs}(t) e^{i2\pi\left(\frac{rx}{2a} + \frac{sy}{2b}\right)} \tag{5-2}$$

$$W_I(x',y',t') = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{W}^{(I)}_{mn}(t') e^{i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} \tag{5-3}$$

$$W_R(x,y,t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \mathcal{W}^{(R)}_{rs}(t) e^{i2\pi\left(\frac{rx}{2a} + \frac{sy}{2b}\right)}, \tag{5-4}$$

where

$$\mathcal{I}_{mn}(t') = \frac{1}{4ab} \int_{-a}^{a} \int_{-b}^{b} I(x',y',t') e^{-i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} dx'\, dy' \tag{5-5}$$

$$\mathcal{R}_{rs}(t) = \frac{1}{4ab} \int_{-a}^{a} \int_{-b}^{b} R(x,y,t) e^{-i2\pi\left(\frac{rx}{2a} + \frac{sy}{2b}\right)} dx\, dy \tag{5-6}$$

$$\mathcal{W}^{(I)}_{mn}(t') = \frac{1}{4ab} \int_{-a}^{a} \int_{-b}^{b} W_I(x',y't') e^{-i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} dx'\, dy' \tag{5-7}$$

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[1] See, for example, Pierre Mertz and Frank Gray

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$$\mathcal{W}^{(R)}_{rs}(t) = \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b} W_R(x,y,t)e^{-i2\pi(\frac{rx}{2a}+\frac{sy}{2b})}dx\,dy. \tag{5-8}$$

Actually the Fourier series (5-1) does not represent just the function $I(x',y',t')$; it also represents the doubly-infinite set of identical functions obtained by displacing the original function $(I(x',y',t')$ through all integral multiples of 2a in the x-direction and of 2b in the y-direction. Hence the line spectrum (5-5) also represents both $I(x',y',t')$ and the doubly-infinite set of functions specified above. Similar statements apply to equations (5-2) through (5-4) and (5-6) through (5-8). In equations (5-5) through (5-8) the lengths a and b must be chosen to satisfy the conditions

$$a \geq a', \quad b \geq b', \qquad \text{(5-9a), (5-9b)}$$

where we have assumed, in accordance with assumption (B), page 3-1 that the dimensions of the replica and the uniform intensity distribution $R_U(t)$ have the same dimensions (2a', 2b') as the original image. These conditions ensure that only one member of the four doubly-infinite sets of functions represented by equations (5-5) through (5-8) is used by restricting the illumination to at most only one member of each of these four sets.

We are not concerned here with errors in reproduction arising from imperfections in the transmission channel, and shall therefore assume that

$$S_R(t) = C_{TR}S_T(t-\tau), \tag{5-10}$$

where the constant C_{TR} is a dimensionless scale factor smaller than unity which represents the fraction of the transmitted power intercepted at the receiver, and where τ is the propagation time.

If we substitute in (5-6) the expression for $R(x,y,t_k)$ given by (2-4), we obtain

$$\mathcal{R}_{rs}(t_k) = \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b}\left\{\int_{t_{k-1}}^{t_k} R_U(t)W_R(x,y,t)dt\right\}e^{-i2\pi(\frac{rx}{2a}+\frac{sy}{2b})}dx\,dy, \tag{5-11}$$

or, interchanging the order of integration,

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$$\mathcal{R}_{rs}(t_k) = \frac{1}{4ab}\int_{t_{k-1}}^{t_k} R_U(t)\left\{\int_{-a}^{a}\int_{-b}^{b} W_R(x,y,t)e^{-i2\pi(\frac{rx}{2a}+\frac{sy}{2b})}dx\,dy\right\}dt, \tag{5-12}$$

which we may in turn, using (5-8), write as

$$\mathcal{R}_{rs}(t_k) = \int_{t_{k-1}}^{t_k} R_U(t)\mathcal{W}_{rs}^{(R)}(t)dt. \tag{5-13}$$

Now, from (2-1) and (2-2), we obtain

$$S_T(t') = C_T\int_{-a'}^{a'}\int_{-b'}^{b'} I(x',y',t')W_I(x',y',t')dx'\,dy' \tag{5-14}$$

and, from (5-10) and (2-3)

$$R_U(t) = C_R C_{TR} S_T(t-\tau). \tag{5-15}$$

Hence, since $(t-\tau)$ is equal to t', (see 2-6), it follows from (5-14) and (5-15) that

$$R_U(t) = C_R C_{TR} C_T\int_{-a'}^{a'}\int_{-b'}^{b'} I(x',y',t-\tau)W_I(x',y',t-\tau)dx'\,dy'. \tag{5-16}$$

Substituting this expression for $R_U(t)$ in (5-13), we obtain

$$\mathcal{R}_{rs}(t_k) = C\int_{t_{k-1}}^{t_k}\left\{\int_{-a}^{a}\int_{-b}^{b} I(x',y',t-\tau)W_I(x',y',t-\tau)dx'\,dy'\right\}\mathcal{W}_{rs}^{(R)}(t)dt \tag{5-17}$$

if we remember (5-9), and that

$$I(x',y',t') \equiv 0 \tag{5-18}$$

for

$$|x'|>a', |y'|>b'; \tag{5-19}$$

and if we write

$$C \equiv C_R C_{RT} C_T. \tag{5-20}$$

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We now express $I(x',y',t-\tau)$ in (5-17) in terms of its two-dimensional line spectrum by using (5-1) and get

$$\mathcal{R}_{rs}(t_k) = C\int_{t_{k-1}}^{t_k} \left\{ \int_{-a}^{a} \int_{-b}^{b} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t-\tau) e^{i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} \mathcal{W}_I(x',y',t-\tau)\,dx'\,dy' \right\} \mathcal{W}_{rs}^{(R)}(t)\,dt, \tag{5-21}$$

which, may be written, in the form

$$\mathcal{R}_{rs}(t_k) = C \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{t_{k-1}}^{t_k} \left\{ \int_{-a}^{a} \int_{-b}^{b} \mathcal{I}_{mn}(t-\tau) e^{i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} \mathcal{W}_I(x',y',t-\tau)\,dx'\,dy' \right\} \mathcal{W}_{rs}^{(R)}(t)\,dt \tag{5-22}$$

by interchanging the order of the summations and integrations. Furthermore, because of Assumption (2-7), we also have

$$\mathcal{I}_{mn}(t-\tau) = \mathcal{I}_{mn}(t_{k-1}-\tau) \qquad \text{for} \qquad t_{k-1} \le t < t_k. \tag{5-23}$$

Hence (5-22) may be written in the form

$$\mathcal{R}_{rs}(t_k) = C \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t_{k-1}-\tau) \int_{t_{k-1}}^{t_k} \left\{ \int_{-a}^{a} \int_{-b}^{b} W_I(x',y',t-\tau) e^{i2\pi\left(\frac{mx'}{2a} + \frac{ny'}{2b}\right)} dx'\,dy' \right\} dt\, \mathcal{W}_{rs}^{(R)}(t)\,dt \tag{5-24}$$

or, because of (5-7), in the form

$$\mathcal{R}_{rs}(t_k) = 4abC \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t_{k-1}-\tau) \int_{t_{k-1}}^{t_k} \mathcal{W}_{mn}^{(I)*}(t-\tau)\, \mathcal{W}_{rs}^{(R)}(t)\,dt, \tag{5-25}$$

where $\mathcal{W}_{mn}^{(I)*}(t-\tau)$ denotes the complex conjugate of $\mathcal{W}_{mn}^{(I)}(t-\tau)$.

By examining this expression — which relates the space spectrum of the replica $R(x,y,t_k)$ (at time t_k) to that of the original image $I(x,y,t_{k-1}-\tau)$ (at the corresponding time $(t_{k-1}-\tau)$ — we can derive sufficient conditions which the weighting functions must satisfy, for this replica to be perfect. These conditions are

$$\int_{t_{k-1}}^{t_k} W^{(I)*}_{mn}(t-\tau)W^{(R)}_{rs}(t)dt = 0 \tag{5-26a}$$

for all t_k and for

$$m \neq r \text{ and/or } n \neq s; \tag{5-26b}$$

and

$$\int_{t_{k-1}}^{t_k} W^{(I)*}_{mn}(t-\tau)W^{(R)}_{rs}(t)dt = \frac{1}{4abC} \tag{5-27a}$$

for all t_k and for

$$m = r,\ n = s. \tag{5-27b}$$

The first condition ensures that only one spectral line of the original image $I(x',y',t')$ contributes to any one spectral line of the replica. The second condition ensures that corresponding spectral lines of the original image and the replica are of equal intensity. We shall refer to these two conditions — in all cases considered in this report — as Conditions I and II respectively. They can be expressed in more compact form by using Kronecker deltas. We have

$$\int_{t_{k-1}}^{t_k} W^{(I)*}_{mn}(t-\tau)W^{(R)}_{rs}(t)dt = \frac{1}{4abC}\delta_{mr}\delta_{rs}, \tag{5-28}$$

which contains both the above conditions in a single equation.

When Conditions I and II are satisfied, we obtain from (5-25)

$$R_{rs}(t_k) = I_{rs}(t_{k-1}-\tau) \tag{5-29}$$

for all t_k; that is, the space spectra of the successive replicas are identical to those of the corresponding original images. Hence we see that these replicas are, in effect, perfect reproductions of these images when Conditions I and II are satisfied.

Conditions (5-26) and (5-27) have been derived by using only the restrictive Assumptions (A) and (B) on page 3-1. They therefore apply to the most general form of weighting functions for which the Fourier series provide a valid representation of these functions and of the original image and its replica in the intervals 2a and 2b. We can distinguish two methods of instrumenting such functions.

(1) The transmissivity patterns are fixed with respect to the original image $I(x',y',t')$ and the uniform intensity $R_U(t)$, but vary with time[1].

(2) The transmissivity patterns are moved across (scan) $I(x',y',t')$ and $R_U(t)$, but are time-invariant.

We shall, in accordance with Assumption C, restrict ourselves to the second method. In this case the distance traversed by the transmissivity patterns during the integration interval T must not be less than 2a', that is,

$$vT \geq 2a', \tag{5-30}$$

where v is the speed of translation of the two transmissivity patterns corresponding to the weighting functions $W_I(x',y',t')$ and $W_R(x,y,t)$. These functions now take the form $W_I(x'-vt',y')$ and $W_R(x-vt,y)$, respectively. Expression (5-8) for $\mathcal{W}_{rs}^{(R)}(t)$ now becomes

$$\mathcal{W}_{rs}^{(R)}(t) = \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b}\mathcal{W}_R(x-vt,y)e^{-i2\pi\left(\frac{rx}{2a}+\frac{sy}{2b}\right)}dx\,dy, \tag{5-31}$$

which, if we make the change of variable

$$X = x-vt \tag{5-32}$$

and introduce Assumption (D), can in turn be written as

$$\mathcal{W}_{rs}^{(R)}(t) = \frac{1}{4ab}\int_{-a-vt}^{a-vt}\int_{-b}^{b}W_R(X,y)e^{-i2\pi\left[\frac{r}{2a}(X+vt)+\frac{sy}{2b}\right]}dX\,dy \tag{5-33}$$

or

$$\mathcal{W}_{rs}^{(R)}(t) = \frac{1}{4ab}e^{-i2\pi\frac{rv}{2a}t}\int_{-a}^{a}\int_{-b}^{b}W_R(X,y)e^{-i2\pi\left(\frac{rX}{2a}+\frac{sy}{2b}\right)}dX\,dy \tag{5-34}$$

[1] That is, either the shape or degree of transparency of these patterns varies with time.

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We have changed in (5-34) the limits of integration of the dummy variable X without changing the value of the integral because $W_R(X,y)$ is by Assumption D, periodic in X, and because the exponential with imaginary exponent is, by definition, periodic in X.

We now write (5-34) in the form

$$\mathcal{W}_{rs}^{(R)}(t) = \mathcal{W}_{rs}^{(R)} e^{-i2\pi\frac{rv}{2a}t}. \tag{5-35}$$

where

$$\mathcal{W}_{rs}^{(R)} \equiv \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b} W_R(X,y) e^{-i2\pi\left(\frac{rX}{2a}+\frac{sy}{2b}\right)} dX\, dy \tag{5-36a}$$

or, since X is only a dummy variable,

$$\mathcal{W}_{rs}^{(R)} \equiv \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b} W_R(x,y) e^{-i2\pi\left(\frac{rx}{2a}+\frac{sy}{2b}\right)} dx\, dy. \tag{5-36b}$$

Similarly,

$$\mathcal{W}_{mn}^{(I)*}(t') = \mathcal{W}_{mn}^{(I)*}(t-\tau) = \mathcal{W}_{mn}^{(I)*} e^{i2\pi\frac{mv}{2a}(t-\tau)} \tag{5-37}$$

or

$$\mathcal{W}_{mn}^{(I)*}(t') = \mathcal{W}_{mn}^{(I)*}(t-\tau) = e^{-i2\pi\frac{mv}{2a}\tau}\mathcal{W}_{mn}^{(I)*} e^{i2\pi\frac{mv}{2a}t}. \tag{5-38}$$

Thus Conditions I and II above now take the form

$$\int_{t_{k-1}}^{t_k} \mathcal{W}_{mn}^{(I)*}\mathcal{W}_{rs}^{(R)} e^{i2\pi\frac{v}{2a}(m-r)t} dt = 0 \tag{5-39a}$$

for

$$m \neq r \text{ and/or } n \neq s, \tag{5-39b}$$

and

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$$\int_{t_{k-1}}^{t_k} W_{mn}^{(I)*} W_{rs}^{(R)} e^{i2\pi \frac{v}{2a}(m-r)t} dt = \frac{e^{i2\pi \frac{mv}{2a}\tau}}{4abC} \tag{5-40a}$$

for

$$m = r \text{ and } n = s \tag{5-40b}$$

Now the equation

$$\int_{t_{k-1}}^{t_k} e^{i2\pi qt} dt = 0 \tag{5-41a}$$

is true whenever

$$q = \frac{K}{t_k - t_{k-1}}, \tag{5-41b}$$

where K is a real non-zero integer; that is, whenever the integration interval is equal to an integral number of periods of the integrand. Also

$$\int_{t_{k-1}}^{t_k} e^{i2\pi qt} dt = t_k - t_{k-1} = T \tag{5-42a}$$

whenever

$$q = 0. \tag{5-42b}$$

The integrals in (5-39a) and (5-40a) have the form of the integral in (5-41a) and (5-42a) if

$$q \equiv \frac{v}{2a}(m-r). \tag{5-43}$$

If we choose

$$2a = v(t_k - t_{k-1}) = vT, \tag{5-44}$$

we have

$$q \equiv \frac{v}{2a}(m-r) = \frac{(m-r)}{t_k - t_{k-1}}. \tag{5-45}$$

Relations (5-39) and (5-40) now become

$$\mathcal{W}_{rn}^{(I)*}\mathcal{W}_{rs}^{(R)} = 0 \tag{5-46a}$$

for

$$n \neq s, \tag{5-46b}$$

and

$$\mathcal{W}_{rn}^{(I)*}\mathcal{W}_{rs}^{(R)} = \frac{e^{i2\pi\frac{rv}{2a}\tau}}{4abCT} \tag{5-47a}$$

for

$$n = s. \tag{5-47b}$$

Equations (5-46) and (5-47) are the forms taken by Conditions I and II in the case of moving invariant transmissivity patterns. They can also be expressed by the single equation

$$\mathcal{W}_{rn}^{(I)*}\mathcal{W}_{rs}^{(R)} = \frac{e^{i2\pi\frac{rv}{2a}\tau}}{4abCT}\,\delta_{ns}. \tag{5-48}$$

We now consider the case where (a) the transmissivity pattern of the two weighting functions are identical and (b) the motions of these two patterns are synchronized so that

$$W_R(x-vt,y) = W_I(x'-v\{t'+\tau\}, y'). \tag{5-49}$$

It follows, since from (2-6)

$$t' = t-\tau, \tag{5-50}$$

that

$$W_R(x-vt,y) = W_I(x'-vt,y'). \tag{5-51}$$

Hence we have, instead of equations (5-37) and (5-38),

$$\mathcal{W}_{mn}^{(I)*}(t'+\tau) = \mathcal{W}_{mn}^{(I)*}(t) = \mathcal{W}_{mn}^{(I)*}e^{i2\pi\frac{mv}{2a}t}. \tag{5-52}$$

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Therefore the conditions, corresponding to conditions (5-46) and (5-47) — when the weighting functions are synchronized according to (5-49) — are

$$W^{(I)*}_{rn} W^{(R)}_{rs} = 0 \tag{5-53a}$$

for

$$n \neq s \tag{5-53b}$$

and

$$W^{(I)*}_{rn} W^{(R)}_{rs} = \left| W_{rs} \right|^2 = \frac{1}{4abCT} \tag{5-54a}$$

for

$$n = s, \tag{5-54b}$$

where

$$W_{rs} = W^{(R)}_{rs} = W^{(I)}_{rs}. \tag{5-55}$$

Conditions I and II can also in this case be represented by the single equation

$$W^{(I)*}_{rn} W^{(R)}_{rs} = \left| W_{rs} \right|^2 \delta_{ns} = \frac{1}{4abCT} \delta_{ns}. \tag{5-56}$$

Note that these conditions also apply to the particular case when 2b is equal to 2b'.

Conditions (5-53) and (5-54) are sufficient to ensure that the weighting functions provide perfect replication under the restrictions imposed by Assumptions (A) through (E). However — as we shall see later — they are not necessary conditions.

5.2 Proof That No Weighting Function Can Satisfy the Sufficient Conditions

We now seek to discover whether there exist spatial line spectra that satisfy conditions (5-53) and (5-54).

The quantities $W^{(I)}_{mn}$ and $W^{(R)}_{rs}$ are the (time-invariant) line spectra of the transmissivity patterns of the weighting functions $W_I(x'-vt,y')$ and $W_R(x-vt,y)$. We note that the first subscripts of these two line spectra are the same in equations (5-53) and (5-54). Thus these conditions impose no constraints on the relation between transmissivity-pattern line spectra whose first subscripts differ.

To discover the implications of the constraints imposed by condition (5-53) on $W^{(I)}_{mn}$ and $W^{(R)}_{rs}$ when their first subscripts are the same (i.e. when m=r), let, for example, the second transmissivity pattern have a non-zero spectral line $W^{(R)}_{r_1 s_1}$ for the specific values

$$r = r_1 \qquad s = s_1 . \tag{5-57}$$

Then, we note from condition (5-53) that all the spectral lines $W^{(I)*}_{r_1 s}$ — and hence also all $W^{(I)}_{r_1 s}$ —, must be zero if they have the same first subscript r_1 and a different second subscript s. Since the two line spectra are equal (see 5-55), it follows that at most one spectral line with any given first subscript (say $r = r_1$) can be non-zero.

We next note that condition (5-54) requires the square of the absolute value of two spectral lines with the same subscripts, say

$$r = r_1 \text{ and } s = s_1 , \tag{5-58}$$

to be non-zero and equal to a (real) number which is independent of the subscripts r and s. Typical line spectra satisfying condition (5-53) are shown in Figure 2a.

Clearly conditions (5-53) and (5-54) cannot both be fulfilled, except for the trivial case when all spectral lines are zero. In effect, the first condition requires some spectral lines to be zero; and this violates the second condition. We shall sometimes refer to the first condition as the "orthogonality" condition.

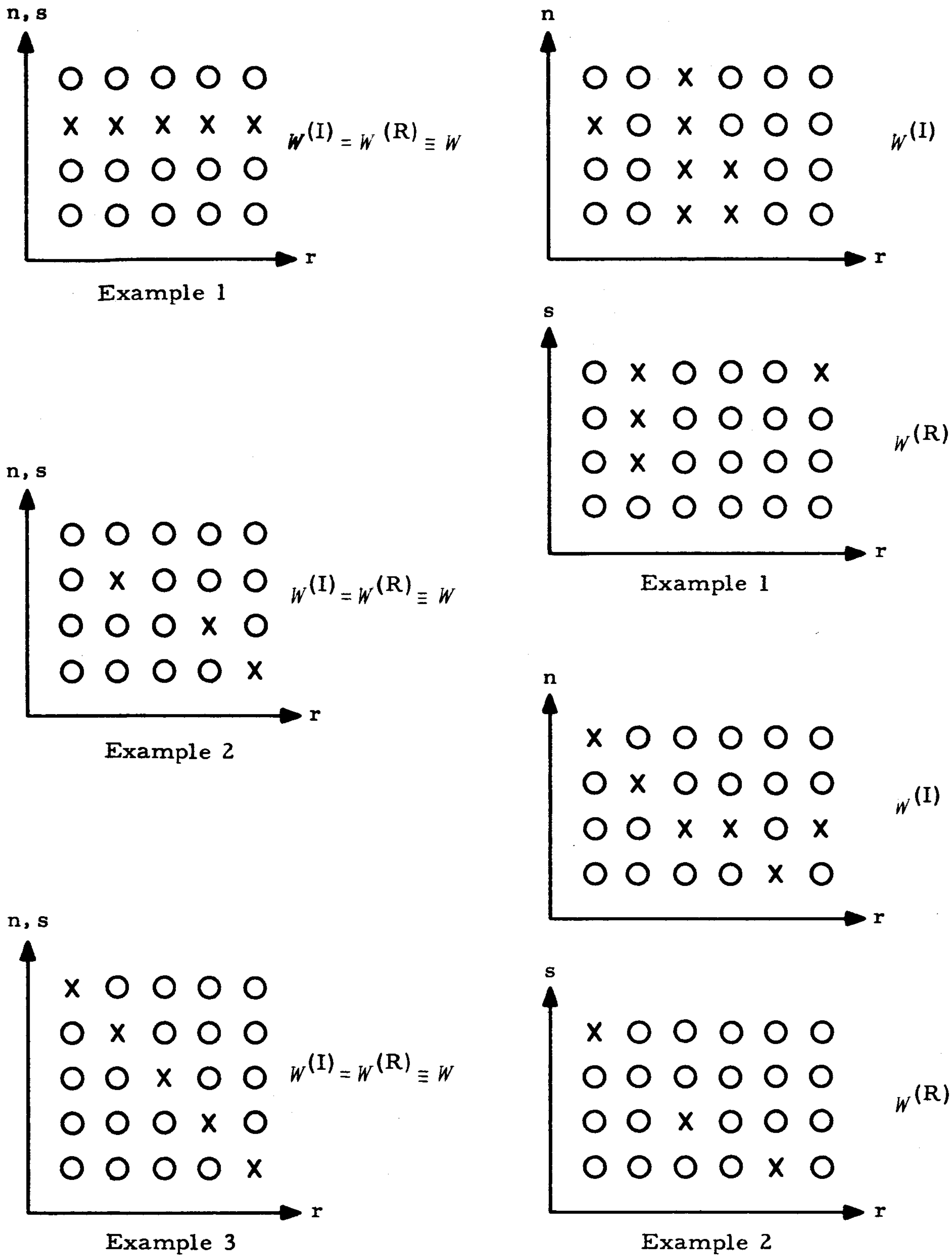

Figure 2a

Figure 2b

Legend

X indicates non-zero spectra

O indicates zero spectra

Note:
Only a few spectral lines are shown

Figure 2. Spectral Pattern Satisfying Condition I (The "Orthogonality" Condition)

We note, parenthetically, that the conditions for perfect replication formulated in Subsection 2.1 cannot be satisfied even when the two transmissivity patterns are different. In effect, a similar argument to that given for condition (5-53) shows that condition (5-46) requires some spectral lines of at least one of the two transmissivity patterns to be zero. Consequently, some of the products $W_{rs}^{(I)*}W_{rs}^{(R)}$ must also be zero; and this violates condition (5-47). Typical line spectra satisfying condition (5-46) are shown in Figure 2b.

5.3 Derivation of Less Restrictive Sufficient Conditions

As mentioned in Subsection 5.1, conditions (5-53) and (5-54) are sufficient but not necessary conditions for perfect replication under Assumptions (A) through (E). These conditions are not necessary because they were derived by:

(a) representing the four functions $I(x',y',t')$, $W_I(x',y',t')$, $W_R(x,y,t)$, and $R(x,y,t)$, and hence also their transmissivity patterns, in terms of Fourier series.

(b) choosing $\{$see (5-44)$\}$ the "space integration interval" vT equal to the fundamental (space) period 2a — in the x-direction — of the transmissivity patterns[1]

(c) choosing, in the x-direction, the basic interval (of expansion) of the four two-dimensional Fourier series equal to the fundamental period 2a of the transmissivity pattern in this direction, and by

(d) choosing not to take advantage of the possibility that the weighting functions can be required to have "internal periodicities", as described later.

[1] We have hitherto used the symbol 2a to denote both the fundamental period of the moving transmissivity patterns and the interval of expansion of the four two-dimensional Fourier series. We shall in the future have occasion to use Fourier series expansion with basic intervals different from the fundamental period of the moving transmissivity patterns. The symbol 2a'' will be used for such intervals.

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These choices are equivalent to additional restrictive assumptions. The implications of assumptions (b) and (c) are now discussed in detail; some of the implications of the other assumptions are discussed in other subsections.

The functions $I(x',y',t')$ and $R(x,y,t)$ are uniquely specified in the interval $2a$ along the x-axis and the interval $2b$ along the y-axis by the Fourier series (5-1) and (5-2) because we have chosen the fundamental periods of each of these two series equal to the same two intervals $2a$ and $2b$. However, we only need these functions to be specified in the intervals $2a'$ and $2b'$, and hence we could also specify $I(x',y',t')$ and $R(x,y,t)$ by Fourier series with fundamental periods equal to $2a'$ and $2b'$. Since intensity distributions are in practice always bandlimited, it follows that fewer spectral lines (Fourier coefficients) are required to determine uniquely $I(x',y',t')$ and $R(x,y,t)$ when

$$a = a' \quad \text{and} \quad b = b' \tag{5-59}$$

than when

$$a > a' \quad \text{and} \quad b > b'. \tag{5-60}$$

Further, because of the assumptions made under (b) and (c) above, it follows that the distance traversed by the transmissivity patterns during the interval of integration T is equal to the chosen common fundamental period, $2a$, for the four functions $I(x',y',t')$, $R(x,y,t)$, $W_I(x',y',t')$, and $W_R(x,y,t)$.

Consequently, if we

(1) choose $2a$ larger than $2a'$, we overspecify the functions $I(x',y',t')$ and $R(x,y,t)$, and if we,

(2) choose $2a$ equal to $2a'$, we limit the distance traversed by the transmissivity patterns during integration to the common width $2a'$ of the original image and its replica.

The above discussion suggests that we should consider the possibility of obtaining perfect replicas by using weighting functions $W_I(x'-vt,y')$ and $W_R(x-vt,y)$ having transmissivity patterns $W_I(x',y')$ and $W_R(x,y)$ formed with sets of patterns whose members have the following properties:

(1) Each member $M_{I,j}(x',y')$ and $M_{R,k}(x,y)$ of $W_I(x',y')$ and $W_R(x,y)$ respectively, has a length $2a''$ and a width $2b'$ and can therefore

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be represented by a Fourier series with the same fundamental periods $2a''$ and $2b'$. This corresponds to removing condition (c).

(2) The patterns $M_{I,j}(x',y')$ and $M_{R,k}(x,y)$ are the same when j is equal to k.[1]

Because of (2) above, we may therefore write

$$M_{I,j}(x',y') = M_{R,j}(x,y) = M_j(x',y') = M_j(x,y). \tag{5-61}$$

The transmissivity patterns $W_I(x',y')$ and $W_R(x,y)$ are formed by using a number, say J, of member-patterns $M_j(x',y')$ and $M_j(x,y)$, respectively, with common boundaries parallel to the y-axis (see Figure 3). We shall refer to the two former transmissivity patterns as W-patterns and to the two latter transmissivity patterns as M-patterns. The transparent areas of two adjacent M-patterns are separated by a distance $2a'$ to ensure that a transparent area of one such member is not illuminated at the same time as that of an adjacent member. Each M-pattern has a length $2a''$ and a width $2b'$, where

$$2a'' = 2a' + d_e \tag{5-62}$$

with

$$d_e = d_t \tag{5-63}$$

where d_t is the length of the region containing transparent areas and d_e is the excess length of M-patterns over the length, $2a'$, of the common aperture of the original image and its replica. The integration interval T is related to $2a''$ and to the speed v of translation by

$$2a''J = vT; \tag{5-64}$$

and the replica obtained at the end of this interval is — since the operations are linear — the sum of the successive replicas $R_j(x,y,t_k)$ obtained by moving the corresponding pair of patterns $M_j(x',y')$ and $M_j(x,y)$ across the original image $I(x',y',t_{k-1})$ and the uniform illumination $R_U(t_k)$; that is

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[1] We may attribute this property to the patterns $M_{I,j}(x',y')$ and $M_{R,k}(x,y)$ because by assumption, the patterns $W_I(x',y')$ and $W_R(x,y)$ are the same.

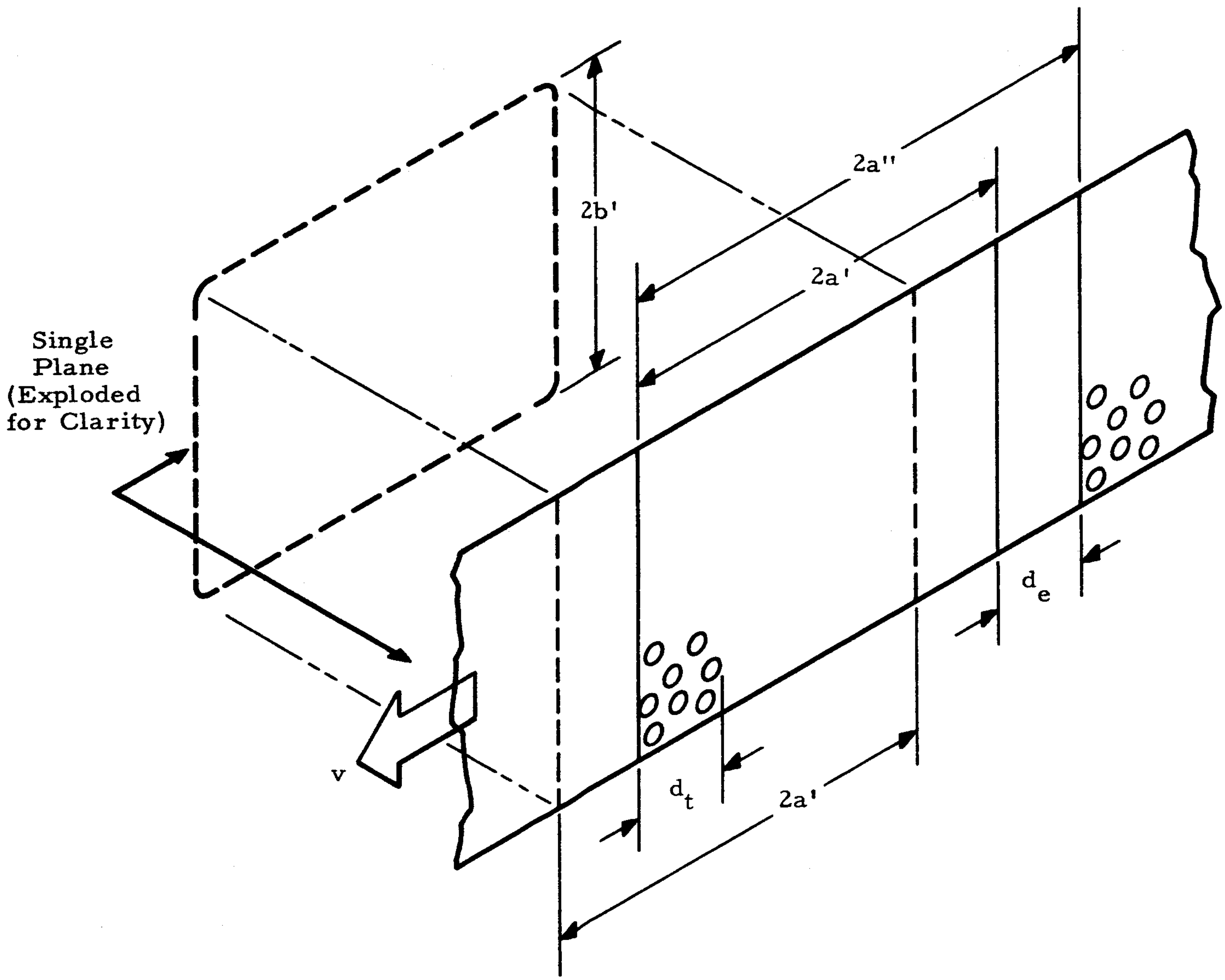

Figure 3. Transmissivity Pattern Satisfying the First Set of Less Restrictive Conditions

$$R(x,y,t_k) = \sum_{j=1}^{J} R_j(x,y,t_k). \tag{5-65}$$

The expressions for $W_{mn}^{(I)*}(t')$ and $W_{rs}^{(R)}(t)$ given by (5-38) and (5-35) are the particular forms taken by the weighting functions when they are instrumented by moving (translating) time-invariant W-patterns. By using these expressions in (5-25), we obtain

$$R_{rs}(t_k) = 4abC \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t_{k-1}-\tau) e^{-2\pi \frac{mv}{2a}\tau} W_{mn}^{(I)*} W_{rs}^{(R)} \int_{t_{k-1}}^{t_k} e^{i2\pi \frac{v}{2a}(m-r)t} dt. \tag{5-66}$$

But, because of (5-41) through (5-45),

$$\int_{t_{k-1}}^{t_k} e^{i2\pi \frac{v}{2a}(m-r)t} dt = T\delta_{mr}, \tag{5-67}$$

and hence

$$\mathcal{R}_{rs}(t_k) = 4abCT \sum_{n=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau) e^{-i2\pi \frac{rv}{2a}\tau} W_{rn}^{(I)*} W_{rs}^{(R)}. \tag{5-68}$$

The synchronization of the weighting functions may be chosen (see equations (5-49) through (5-52) so as to eliminate the exponential factor in (5-68).[2] We shall therefore write

$$\mathcal{R}_{rs}(t_k) = 4abCT \sum_{n=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau) W_{rn}^{(I)*} W_{rs}^{(R)}. \tag{5-69}$$

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[2] This is true even if the transmissivity patterns of the two weighting functions are different.

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Because the separation between transparent areas of any two adjacent M-patterns has been chosen large enough for light to get through only one M-pattern at any given time, we can assume that each M-pattern is part of a periodic pattern in the x-direction consisting of contiguous identical patterns, and that we use only one member of each such periodic pattern in forming the replica $R_J(x,y,t_k)$. Let $R_j(x,y,t_k)$ be the contribution to $R(x,y,t_k)$ obtained by moving a pair of identical M-patterns across the original image $I(x',y',t)$ and the uniform illumination $R_U(t)$. Then, because we may assume each M-pattern to be a single period of a doubly-periodic pattern, we may, by analogy with (5-66) (which holds when the basic interval (of expansion) 2a for all four two-dimensional Fourier series is equal to the fundamental period of the transmissivity patterns[2]), write

$$\mathcal{R}_{j,rs}(t_k) = 4a''b'CT_J \sum_{n=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau)\mathcal{M}^{(I)*}_{j,rn}\mathcal{M}^{(R)}_{j,rs} \; ; \tag{5-70}$$

where $\mathcal{R}_{j,rs}(t_k)$, $\mathcal{M}^{(I)}_{j,rn}$, and $\mathcal{M}^{(R)}_{j,rs}$ represent the spatial line spectra of $R_j(x,y,t_k)$, $M^{(I)}_j(x',y')$, and $M^{(R)}_j(x,y)$, respectively, and of their corresponding three doubly-infinite sets of identical functions obtained by displacing the original functions through all integral multiple values of 2a'' in the x-direction and of 2b' in the y-direction; and where

$$T_J = \frac{T}{J} \tag{5-71}$$

It follows from (5-65) — since Fourier series transforms are linear —, that

$$\mathcal{R}_{rs}(t_k) = \sum_{j=1}^{J} \mathcal{R}_{j,rs}(t_k) \tag{5-72}$$

Consequently, substituting in this relation the expression of $\mathcal{R}_{j,rs}(t_k)$ given by (5-70), we obtain (at the instant t_k)

$$\mathcal{R}_{rs}(t_k) = 4a''b'CT_J \sum_{n=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau) \sum_{j=1}^{J} \mathcal{M}^{(I)*}_{j,rn}\mathcal{M}^{(R)}_{j,rs}. \tag{5-73}$$

To obtain a perfect replica we must have

$$\mathcal{R}_{rs}(t_k) = \mathcal{I}_{rs}(t_{k-1}-\tau), \tag{5-74}$$

and this relation is true if

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[2] Note that, although the fundamental period of individual M-patterns is assumed (for the purpose of expansion) to be 2a'', the fundamental period of the W-patterns is 2a.

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$$\sum_{j=1}^{J} M^{(I)*}_{j,rn} M^{(R)}_{j,rs} = 0 \quad (5\text{-}75a)$$

for

$$n \neq s, \quad (5\text{-}75b)$$

and if

$$\sum_{j=1}^{J} M^{(I)*}_{j,rn} M^{(R)}_{j,rs} = \sum_{j=1}^{J} \left| M_{j,rs} \right|^2 = \frac{1}{4a''b'CT_J} \quad (5\text{-}76a)$$

for

$$n = s. \quad (5\text{-}76b)$$

Conditions (5-75) and (5-76) correspond to conditions (5-53) and (5-54) in the case of the M_J-patterns. They can also be expressed by a single equation, that is, by

$$\sum_{J=1}^{J} M^{(I)*}_{j,rn} M^{(R)}_{j,rs} = \frac{1}{4a''b'CT_J} \delta_{ns}. \quad (5\text{-}77)$$

These conditions for the M-patterns are less restrictive than those for the W-patterns because the former conditions impose only a single condition on the sum of the M-patterns and not on each individual M-pattern.

5.4 General Form of Physically Realizable Weighting Functions Satisfying the First Set of Less Restrictive Conditions

We shall now show that there exist M-patterns which satisfy the conditions (5-75) and (5-76) if — as is always true in practice — the relevant functions are bandlimited. To this end, we consider the case when the members $M_J(x',y')$ and $M_J(x,y)$ of $W_I(x',y)$ and $W_R(x,y)$, respectively, consist of identical transmissivity patterns that are displaced successively through distance $2b'/J$ parallel to the direction normal to the velocity of $\underline{v}$ of translation. That is, we write

$$M_J(x',y') = M(x',y'-j\frac{2b'}{J}) \quad (j=1,2,\ldots J) \tag{5-78}$$

and

$$M_J(x,y) = M(x,y-j\frac{2b'}{J}) \quad (j=1,2,\ldots J) \tag{5-79}$$

It follows that the spatial line spectrum $M_{J,mn}$ of either of these functions is given by

$$\mathcal{M}_{j,mn} = \mathcal{M}_{mn} e^{-i2\pi\frac{n}{2b'}\cdot j\frac{2b'}{J}} = \mathcal{M}_{mn} e^{-i2\pi j\frac{n}{J}} \tag{5-80}$$

We first show that a W-pattern consisting of J such M-patterns satisfies condition (5-75).

Because

$$\sum_{j=1}^{J} \mathcal{M}^{(I)*}_{j,rn} \mathcal{M}^{(R)}_{j,rs} = \mathcal{M}^{*}_{rn} \mathcal{M}_{rs} \sum_{j=1}^{J} e^{i2\pi j\frac{n-s}{J}} \tag{5-81}$$

and because

$$\sum_{j=1}^{J} e^{i2\pi j\frac{n-s}{J}} = 0 \tag{5-82a}$$

for

$$n \neq s+KJ \tag{5-82b}$$

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where K is an integer or zero, it follows that condition (5-75) is identically satisfied for all such values of n. We therefore need only consider whether this condition can be satisfied for values of n given by

$$n = s+KJ \quad (K \neq 0). \tag{5-83}$$

We shall now prove that (5-75) can be satisfied for such values of n provided the functions $I(x',y')$, $M(x',y')$, and $M(x,y)$ are bandlimited in a direction normal to the velocity $\underline{v}$ of translation. (These functions need not be bandlimited in a direction parallel to $\underline{v}$.) Let

$$\mathcal{I}_{mn} = 0 \qquad \text{for} \qquad |n| > N \tag{5-84}$$

and

$$\mathcal{M}_{rs} = 0 \qquad \text{for} \qquad |s| > S, \tag{5-85}$$

where N and S are positive integers. Since physically realizable functions are always effectively bandlimited, there always exist an N and S for which these assumptions are true. (A necessary condition for the original image $I(x',y',t_k)$ not to be degraded is that

$$S \geq N.) \tag{5-86}$$

Then, if

$$J > 2S, \tag{5-87}$$

condition (5-75) is always satisfied because, on account of relation (5-83), either

$$|s| > S \tag{5-88a}$$

and/or

$$|n| > S \tag{5-88b}$$

for all

$$K \neq 0. \tag{5-88c}$$

Hence at least one of the two factors $\mathcal{M}^*_{rn}$ and $\mathcal{M}_{rs}$ in (5-81) vanishes, and consequently condition (5-75) is satisfied.

Condition (5-87) was obtained by trying to satisfy condition (5-75). If, however, note that the weaker condition[1]

$$J > N+S \tag{5-89}$$

is sufficient to ensure perfect replication when the M-patterns have the general form given by (5-78) and (5-79). We show them as follows.

Because of (5-84), we are interested only in cases when

$$|n| \le N. \tag{5-90}$$

In these cases, since from (5-83)

$$s-n = -KJ \tag{5-91}$$

and consequently

$$|s-n| = |KJ|, \tag{5-92}$$

it follows that

$$|s-n| > N+S \tag{5-93}$$

when

$$J > N+S \text{ and } K \neq 0. \tag{5-89}$$

But

$$|s-n| \le |s| + |n|,$$

and therefore, using (5-93),

$$N+S < |s-n| \le |s| + |n| \tag{5-94}$$

so that

$$N+S < |s| + |n| \tag{5-95}$$

or

$$|s| > N+S - |n|. \tag{5-96}$$

Moreover, from (5-90)

$$N - |n| \ge 0, \tag{5-97}$$

so that, substituting in (5-96), we obtain

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[1] Compared to condition (5-84).

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$$|s| > N+S - |n| \geq S. \tag{5-98}$$

or

$$|s| > S. \tag{5-99}$$

For values of S satisfying this last inequality, all the terms on the right-hand side of (5-73) vanish and consequently condition (5-89) is in fact sufficient to ensure perfect replication.

We note by examining (5-73) that if the resolution of the integrating screen, on which the replica is displayed, is limited so that

$$\mathcal{R}_{rs} = 0 \qquad \text{for} \qquad |s| > S, \tag{5-100}$$

the same value of J will ensure that a spectral line of the original image contributes only to the proper spectral line of the replica. This follows by an argument similar to that used above since the limited resolution of the integrating screen has an effect equivalent to that of the limited bandwidth of $M^{(R)}(x,y)$.

We next note that

$$\sum_{j=1}^{J} e^{i2\pi j\frac{n-s}{j}} = J \tag{5-101a}$$

for

$$n = s, \tag{5-101b}$$

so that, because of (5-81) and (5-71), condition (5-76) becomes

$$M_{rn}^{(I)*} M_{rs}^{(R)} = \left| M_{rs} \right|^2 = \frac{1}{4a''b''CT} \tag{5-102a}$$

for

$$n = s. \tag{5-102b}$$

This condition can always be satisfied, in principle, provided the original image is bandlimited, that is provided

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$$I_{mn} = 0 \tag{5-103a}$$

for

$$m > M \text{ and/or } n > N, \tag{5-103b}$$

where M and N are finite positive integers. It can also always be satisfied in practice for all spectral lines of interest, that is, for all spectral lines I_{mn} of a physically relizable image that contain significant intensity. This subject is discussed in detail in Section 6.

5.5 Second Set of Less Restrictive Conditions

In Subsection 5.3 we derived less restrictive conditions for the transmissivity patterns of the weighting functions than those derived in Subsection 5.1. This was achieved by removing the restrictive assumption (c) given in Subsection 5.2. We now seek to discover an alternative set of less restrictive conditions on the transmissivity patterns by removing the restrictive assumption (d).

We choose

(1) the basic intervals (of expansion) of the Fourier series representing the original image, its replica, and the transmissivity patterns, equal

(i) in the x-direction, to the fundamental period 2a of these transmissivity patterns in this direction, and equal

(ii) in the y-direction, to the width 2b' of the original image and its replica.

(2) the basic interval 2a equal to vT;

(3) internal periodicities for the transmissivity patterns in directions not parallel to the x-axis and the y-axis such (see Figure 4) that their fundamental periods have x and y-components equal to 2a(a'/a) and 2b'(a'/a), respectively, while retaining periodicities with a fundamental period equal to vT in the x-direction and a fundamental period equal to 2b' in the y-direction.

(4) the product vT so that the ratio a'/a is an integer J.

or

$$W_{rn}^{(I)} e^{-i2\pi\frac{r+n}{J}} = W_{rn}^{(I)}, \qquad (5\text{-}104b)$$

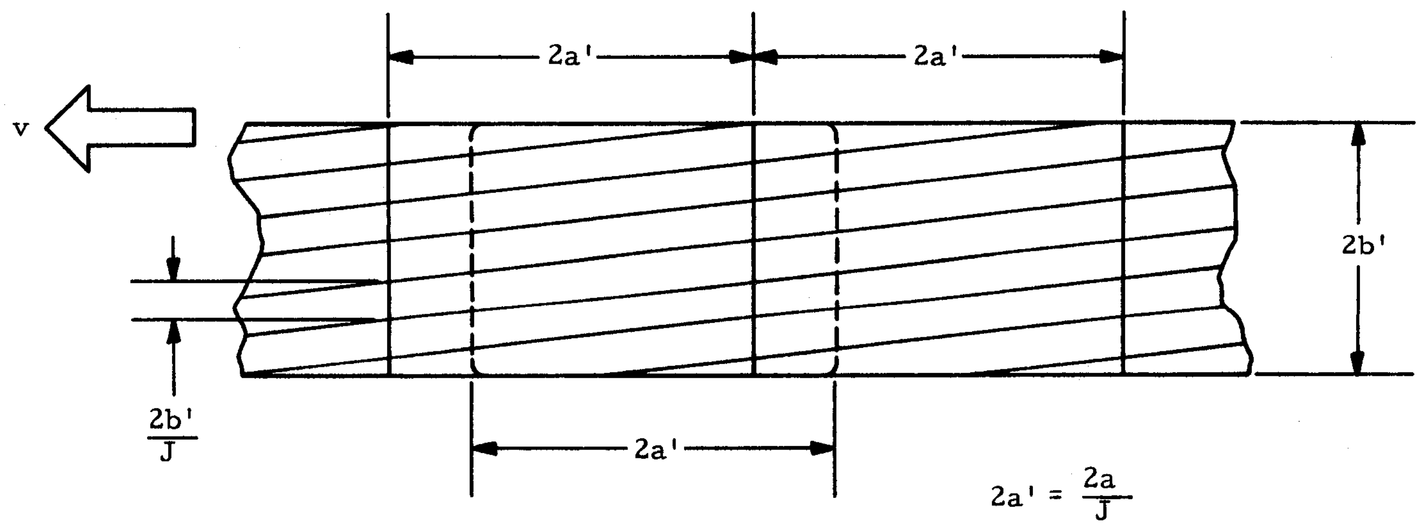

Figure 4. Internal Periodicity of Transmissivity Pattern Satisfying the Second Set of Less Restrictive Conditions.

and

$$W^{(R)}_{rs} e^{-i2\pi\frac{r+s}{J}} = W^{(R)}_{rs}. \tag{5-105}$$

We now assume that I(x′,y′,t′) and R(x,y,t) contain the same internal periodicities inside the region delineated by the basic intervals 2a and 2b′. This assumption, although it in no way affects the form of these functions inside the aperture bounded by

$$x = \pm a' \quad \text{and} \quad y = \pm b', \tag{5-106}$$

does allow one to represent the original image, its replica, and the transmissivity patterns, in terms of the same spectral lines. It also allows us to write, by analogy with (5-94) and (5-95),

$$I_{rn} e^{-i2\pi\frac{r+n}{J}} = I_{rn} \tag{5-107}$$

and

$$R_{rs} e^{-i2\pi\frac{r+s}{J}} = R_{rs}. \tag{5-108}$$

Because

$$e^{-i2\pi\frac{r+n}{J}} = 1 \tag{5-109}$$

only for

$$\frac{r+n}{J} = K_1 \tag{5-110}$$

and because

$$e^{-i2\pi\frac{r+s}{J}} = 1 \tag{5-111}$$

only for

$$\frac{r+s}{J} = K_2, \tag{5-112}$$

where K_1 and K_2 are integers, it follows that we need only consider the values of n and s for which

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$$n = K_1 J - r \tag{5-113}$$

and

$$s = K_2 J - r, \tag{5-114}$$

respectively. This is true because all other spectral lines vanish. That is, the choice of internal periodicity, which leads to equations (5-104) through (5-105), requires that all spectral lines vanish except those for which the exponential factors are equal to unity — that is, those for which K_1 and K_2 are integers. Consequently (5-69) becomes

$$\mathcal{R}_{r,K_2J-r}(t_k) = 4ab'CT \sum_{K_1=-\infty}^{\infty} \mathcal{I}_{r,K_1J-r}(t_{k-1}-\tau) \mathcal{W}^{(I)*}_{r,K_1J-r} \mathcal{W}^{(R)}_{r,K_2J-r}. \tag{5-115}$$

From equation (5-115), it is easy to derive a sufficient condition on the weighting function spectra for perfect replication, namely

$$\mathcal{W}_{r,K_2J-r} \mathcal{W}_{r,K_1J-r} = \frac{\delta_{K_1K_2}}{4ab'CT}. \tag{5-116}$$

As in Subsection 5.4, if

$$\mathcal{I}_{mn} = 0 \qquad \text{for} \qquad n > N \tag{5-117}$$

and

$$\mathcal{M}_{rs} = 0 \qquad \text{for} \qquad s > S \geq N, \tag{5-118}$$

the orthogonality condition can be satisfied by choosing

$$J > 2S. \tag{5-119}$$

The argument leading to the weaker condition

$$J > N+S \tag{5-120}$$

also applies, and so do the comments concerning an integrating screen of limited resolution. It follows that, when J satisfies condition (5-120), all the terms in (5-115) vanish except those for which both K_1 and K_2 are zero. Consequently, (5-115) now becomes

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$$\mathcal{R}_{r,-r}(t_K) = 4ab'CT\, \mathcal{I}_{r,-r}(t_{k-1}-\tau)\mathcal{W}^{(I)*}_{r,-r}\mathcal{W}^{(R)}_{r,-r}, \tag{5-121}$$

and hence the condition for a perfect replica is

$$\mathcal{W}^{(I)*}_{r,-r}\mathcal{W}^{(R)}_{r,-r} = \left|\mathcal{W}_{r,-r}\right|^2 = \frac{1}{4ab'CT}. \tag{5-122}$$

We note that no condition corresponding to the "orthogonality" requirement expressed by Condition I is necessary. We have, in effect succeeded in representing the functions $I(x',y',t')$, $W_I(x',y',t')$, $W_R(x,y,t)$, and $R(x,y,t)$, in such a way that the only non-zero spectral lines lie on a single "diagonal" line in the spectral domain.

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5.6 General Form of Physically Realizable Weighting Functions Satisfying the Second Set of Less Restrictive Conditions

Let M(x,y) and

$$M_j(x,y) = M(x-j\frac{2a}{J}\ ,\ y-j\frac{2b'}{J}) \tag{5-123}$$

be the fundamental functions from which we construct the transmissivity patterns

$$W^{(I)}(x',y') = \sum_{j=1}^{J} M_j(x',y') \tag{5-124a}$$

and

$$W^{(R)}(x,y) = \sum_{j=1}^{J} M_j(x,y) \tag{5-125b}$$

of length

$$2a = vT \tag{5-126}$$

and width 2b'. Hence the spatial spectra of these two functions are both given by expressions of the form

$$\mathcal{W}_{rs} = \mathcal{M}_{rs}\sum_{j=1}^{J} e^{-i2\pi j\frac{r+s}{J}}\ , \tag{5-127}$$

where $\mathcal{M}_{rs}$ represents a Fourier series expansion of M(x,y) over the intervals

$$2a = vT \tag{5-128}$$

(and 2b equal 2b'), and not over the intervals

$$2a'' = \frac{vT}{J}\ . \tag{5-129}$$

Hence

$$\mathcal{W}_{rs} = 0 \qquad \text{for} \qquad s \neq KJ-r \tag{5-130}$$

and

$$\mathcal{W}_{rs} = J\mathcal{M}_{rs} \qquad \text{for} \qquad s = KJ-r, \tag{5-131}$$

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where K is an integer. Consequently the only non-zero terms of W_{rs} are

$$W_{r,KJ-r} = JM_{r,KJ-r}; \tag{5-132}$$

and if upper limits are placed on the spectral lines, as discussed in the previous subsection, the only non-zero terms are

$$W_{r,-r} = JM_{r,-r}. \tag{5-133}$$

It therefore follows from (5-109) that the condition for a perfect replica is

$$M^{*}_{r,-r}M_{r,-r} = \left|M_{r,-r}\right|^2 = \frac{1}{4ab'CTJ}. \tag{5-134}$$

This condition can always be satisfied for physically realizable functions. Similar comments to those made at the end of Subsection 5.4 apply.

We have thus found that physically realizable weighting functions exist that consist of patterns $M_J(x,y)$ which need not contain opaque regions of length d_e to prevent light from illuminating transparent portions of two adjacent patterns simultaneously. In other words, we have proved that there exist transmissivity patterns consisting of transparent areas — as well as of a single transparent spot — that provide perfect replicas.

5.7 Alternative Form of Physically Realizable Weighting Functions Satisfying the First Set of Less Restrictive Conditions

We return, now, to the situation described in 5.3 above, where $W(x,y)$ consists of a set of patterns $M_j(x,y)$ with the several member patterns of the set separated (in the x-direction) by opaque regions of length $2a'$ so that there is no interaction between different member patterns. On this basis, we derived the condition

$$\sum_{j=1}^{J} \mathcal{M}^{(I)*}_{jrn} \mathcal{M}^{(R)}_{jrs} = \frac{\delta_{ns}}{4a''b'CT_J} \quad . \tag{5-135}$$

In 5.4, we described a set of patterns $M_j(x,y)$ which satisfy this condition. We now describe another set. (It is convenient, at this point, to drop the distinction between M_{I_J} and M_{R_J}.) Let

$$M_j(x,y) = \sum_{k=1}^{K} P_{jk}(x,y), \tag{5-136}$$

where

$$P_{jk}(x,y) = P(x-\xi_{jk}, y-\eta_{jk}) \quad . \tag{5-137}$$

Then, expanding $P(x,y)$ over the region $2a''$ by $2b'$,

$$\mathcal{P}_{jkmn} = \mathcal{P}_{mn} e^{-i2\pi\left(\frac{m\xi_{jk}}{2a''} + \frac{n\eta_{jk}}{2b'}\right)} \tag{5-138}$$

and

$$\mathcal{M}_{jmn} = \sum_{k=1}^{K} \mathcal{P}_{jkmn} = \mathcal{P}_{mn} \sum_{k=1}^{K} e^{-i2\pi\left(\frac{m\xi_{jk}}{2a''} + \frac{n\eta_{jk}}{2b'}\right)} \quad . \tag{5-139}$$

We now suppose that the 2JK quantities ξ_{jk} and η_{jk} are statistically independent of each other, and that

(a) the quantities ξ_{jk} are randomly distributed over the interval from $-a''$ to $+a''$, having uniform density with respect to j and k, and

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(b) the quantities η_{jk} are randomly distributed over the interval from -b' to +b', having uniform density with respect to j and k.

Now let

$$\sum_{k=1}^{K} e^{-i2\pi\left(\frac{m\xi_{jk}}{2a''} + \frac{n\eta_{jk}}{2b'}\right)} \equiv E_{jmn}, \tag{5-140}$$

so that

$$M_{jmn} = P_{mn} E_{jmn} \tag{5-141}$$

and

$$M^*_{jrn} M_{jrs} = P^*_{rn} P_{rs} E^*_{jrn} E_{jrs}. \tag{5-142}$$

We now observe that

$$E_{joo} = K. \tag{5-143}$$

For the general case E_{jmn}, we can estimate $\langle |E_{jmn}|^2 \rangle$, the "expected value of" $|E_{jmn}|^2$, on a statistical basis. If K is large, the standard deviation of the estimate will be small compared to the estimate itself. We have

$$\langle |E_{jmn}|^2 \rangle = \left\langle \sum_{k=1}^{K} e^{i2\pi\left(\frac{m\xi_{jk}}{2a''} + \frac{n\eta_{jk}}{2b'}\right)} \sum_{k'=1}^{K} e^{-i2\pi\left(\frac{m\xi_{jk'}}{2a''} + \frac{n\eta_{jk'}}{2b'}\right)} \right\rangle$$

$$= \left\langle \sum_{k=1}^{K} \sum_{k'=1}^{K} e^{i2\pi\left(\frac{m(\xi_{jk}-\xi_{jk'})}{2a''} + \frac{n(\eta_{jk}-\eta_{jk'})}{2b'}\right)} \right\rangle$$

$$= \sum_{k=1}^{K} \sum_{k'=1}^{K} \left\langle e^{i2\pi\left(\frac{m(\xi_{jk}-\xi_{jk'})}{2a''} + \frac{n(\eta_{jk}-\eta_{jk'})}{2b'}\right)} \right\rangle . \tag{5-144}$$

There are K separate terms in the double sum for which

$$k' = k. \tag{5-145}$$

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Each of these reduces to unity. The remaining terms, of which there are $(K^2 - K)$, each take the form

$$\left\langle e^{i2\pi\frac{m\xi_{jk}}{2a''}} e^{-i2\pi\frac{m\xi_{jk'}}{2a''}} e^{i2\pi\frac{n\eta_{jk}}{2b'}} e^{-i2\pi\frac{n\eta_{jk'}}{2b'}} \right\rangle, \quad k' \neq k.$$

Since the factors are statistically **independent,** the expected value of this product is simply

$$\left\langle e^{i2\pi\frac{m\xi_{jk}}{2a''}} \right\rangle \left\langle e^{-i2\pi\frac{m\xi_{jk'}}{2a''}} \right\rangle \left\langle e^{i2\pi\frac{n\eta_{jk}}{2b'}} \right\rangle \left\langle e^{-i2\pi\frac{n\eta_{jk'}}{2b'}} \right\rangle,$$

and all four factors vanish. Accordingly,

$$\left\langle \left| E_{jmn} \right|^2 \right\rangle = K \tag{5-146a}$$

for $\quad m \neq 0 \quad$ and/or $\quad n \neq o.$ (5-146b)

Using these results for the case

$n = s,$

we have

$$M^*_{jrs} M_{jrs} = P^*_{rs} P_{rs} E^*_{jrs} E_{jrs} = P^*_{rs} P_{rs} \left| E_{jrs} \right|^2; \tag{5-147}$$

so that

$$M^*_{joo} M_{joo} = P^*_{oo} P_{oo} K^2 \tag{5-148}$$

and

$$\left\langle M^*_{jrs} M_{jrs} \right\rangle = P^*_{rs} P_{rs} K \tag{5-149a}$$

for $\quad r \neq o \quad$ and/or $\quad s \neq 0.$ (5-149b)

Since $M^*_{joo} M_{joo}$ and $\left\langle M^*_{jrs} M_{jrs} \right\rangle$ are real quantities, we can sum directly over j to form

$$\sum_{j=1}^{J} M^*_{joo} M_{joo} = P^*_{oo} P_{oo} K^2 J, \tag{5-150}$$

and

$$\left\langle \sum_{j=1}^{J} M^*_{jrs} M_{jrs} \right\rangle = P^*_{rs} P_{rs} KJ \tag{5-151a}$$

for $\quad r \neq 0 \quad$ and/or $\quad s \neq 0.$ (5-151b)

For the case

$$n \neq s, \tag{5-152}$$

we have

$$\sum_{j=1}^{J} M^*_{jrn} M_{jrs} = P^*_{rn} P_{rs} \sum_{j=1}^{J} E^*_{jrn} E_{jrs} \tag{5-153}$$

and

$$\left\langle \left| \sum_{j=1}^{J} M^*_{jrn} M_{jrs} \right|^2 \right\rangle = \left| P^*_{rn} P_{rs} \right|^2 \left\langle \left| \sum_{j=1}^{J} E^*_{jrn} E_{jrs} \right|^2 \right\rangle . \tag{5-154}$$

Now

$$\left\langle \left| \sum_{j=1}^{J} E^*_{jrn} E_{jrs} \right|^2 \right\rangle = \left\langle \sum_{j=1}^{J} E^*_{jrn} E_{jrs} \sum_{j'=1}^{J} E_{j'rn} E^*_{j'rs} \right\rangle$$

$$= \left\langle \sum_{j=1}^{J} \sum_{j'=1}^{J} E^*_{jrn} E_{jrs} E_{j'rn} E^*_{j'rs} \right\rangle$$

$$= \sum_{j=1}^{J} \sum_{j'=1}^{J} \left\langle E^*_{jrn} E_{jrs} E_{j'rn} E^*_{j'rs} \right\rangle . \tag{5-155}$$

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In the sum on the right, there are J terms for which

$$J' = J$$

For these, we have

$$\left\langle E^*_{jrn} E_{jrs} E_{jrn} E^*_{jrs} \right\rangle = \left\langle E^*_{jrn} E_{jrn} \right\rangle \left\langle E_{jrs} E^*_{jrs} \right\rangle$$

$$= \left\langle \left|E_{jrn}\right|^2 \right\rangle \left\langle \left|E_{jrs}\right|^2 \right\rangle, \tag{5-156}$$

since E_{jrn} and E_{jrs} are statistically independent quantities when n is different from s. For the remaining terms numbering (J^2-J), we have

$$\left\langle E^*_{jrn} E_{jrs} E_{j'rn} E^*_{j'rs} \right\rangle = \left\langle E^*_{jrn} \right\rangle \left\langle E_{jrs} \right\rangle \left\langle E_{j'rn} \right\rangle \left\langle E^*_{jrs} \right\rangle \tag{5-157}$$

since the four quantities E^*_{jrn}, E_{jrs}, $E_{j'rn}$, and $E^*_{j'rs}$ are statistically independent when n is different from s and j' is different from j. Because n is different from s, at least one of the four quantities E^*_{jrn}, E_{jrs}, $E_{j'rn}$, and $E^*_{j'rs}$ has a third subscript different from zero; consequently at least one of the expected values on the right-hand side of (5-157) is zero. Hence all terms with j' different from j vanish.

This is true because E_{jmn} (where m and/or n is non-zero) isa quantity with a mean-square amplitude of K and a uniformly distributed random phase angle. Hence its expected value is zero.

On the basis of the above argument — which applies when n is different from s — we have

$$\left\langle \left|\sum_{j=1}^{J} M^*_{jrn} M_{jrs}\right|^2 \right\rangle = \left|P^*_{rn} P_{rs}\right|^2 \sum_{j=1}^{J} \left\langle \left|E_{jrn}\right|^2 \right\rangle \left\langle \left|E_{jrs}\right|^2 \right\rangle. \tag{5-158}$$

Hence

$$\left\langle \left|\sum_{J=1}^{J} M^*_{jon} M_{joo}\right|^2 \right\rangle = \left|P^*_{on} P_{oo}\right|^2 K^3 J \qquad \text{for} \qquad n \neq 0, \tag{5-159}$$

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$$\left\langle \left| \sum_{j=1}^{J} M^*_{joo} M_{jos} \right|^2 \right\rangle = \left| P^*_{oo} P_{os} \right|^2 K J^3 \qquad \text{for} \qquad s \neq 0, \tag{5-160}$$

$$\left\langle \left| \sum_{j=1}^{J} M^*_{jrs} M_{jrn} \right|^2 \right\rangle = \left| P^*_{rn} P_{rs} \right|^2 K J^2 \tag{5-161a}$$

for

$$r \neq 0 \qquad \text{and/or} \qquad \begin{cases} n \neq 0 \\ \text{and} \\ s \neq 0. \end{cases} \tag{5-161b}$$

The condition on the M_J's are (see Subsection 5.3)

$$\sum_{j=1}^{J} M^*_{jrn} M_{rs} = \frac{\delta_{ns}}{4a''b'CT_J} \ . \tag{5-77}$$

This may be rewritten as

$$\frac{4a''b'CT}{J} \sum_{j=1}^{J} M^*_{jrn} M_{jrs} = \delta_{ns}. \tag{5-162}$$

We do not attempt to meet the condition in this form, since — in the case considered in this subsection — the left-hand side is only known in a statistical sense. Instead, we substitute the statistically equivalent condition

$$\frac{(4a''b'CT)^2}{J^2} \left\langle \left| \sum_{j=1}^{J} M^*_{jrn} M_{jrs} \right|^2 \right\rangle = \delta_{ns}. \tag{5-163}$$

Using the values for the expected value

$$\left\langle \left| \sum_{j=1}^{J} M^*_{jrn} M_{jrs} \right|^2 \right\rangle$$

given in (5-159) through (5-161), we obtain the five following conditions.

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For the case

$$n \neq s, \tag{5-164}$$

we have the three requirements

$$\frac{(4a''b'CT)^2}{J} \left| P^*_{on} P_{oo} \right|^2 K^3 = 0, \qquad \text{for} \qquad n \neq 0 \tag{5-165}$$

$$\frac{(4a''b'CT)^2}{J} \left| P^*_{oo} P_{os} \right|^2 K^3 = 0, \qquad \text{for} \qquad s \neq 0 \tag{5-166}$$

$$\frac{(4a''b'CT)^2}{J} \left| P^*_{rn} P_{rs} \right| K^2 = 0; \tag{5-167a}$$

for

$$r \neq 0 \qquad \text{and/or} \qquad \begin{cases} n \neq 0 \\ \text{and} \\ s \neq 0\,; \end{cases} \tag{5-167b}$$

and for the case

$$n = s, \tag{5-168}$$

we have the two requirements

$$(4a''b'CT)^2 \left| P^*_{oo} P_{oo} \right|^2 K^4 = 1 \tag{5-169}$$

$$(4a''b'CT)^2 \left| P^*_{rs} P_{rs} \right|^2 K^2 = 1. \tag{5-170}$$

We note that the conditions (5-165) through (5-167) can be satisfied exactly when J is infinite, that is, when the transmissivity patterns are infinitely long. However, we can make the expressions on the left-hand side of these equations as small as we wish by choosing J large enough — and thus also make the unwanted contributions to the replica as small as we wish. The relation between the "noise" produced by these unwanted contributions and the number J is discussed later in this Subsection.

It can be shown that condition (5-169) cannot be satisfied with non-negative transmissivities when K is greater than unity, and hence cannot be satisfied with physically realizable weighting functions. Failure to satisfy this condition results in too high a value of R_{oo}. This corresponds to the superposition of a spatially uniform intensity level on the otherwise perfect replica. This situation can be corrected by using an integrating screen that permits us to subtract this unwanted uniform intensity — for example, by optical quenching — without otherwise affecting the replica. The information needed to control this subtraction is available in electrical form in the receiver.

Librascope has an integrating screen with this property.

We note that a condition equivalent to (5-169) does not appear in our previous discussion on non-random transmissivity patterns. The reason for this fact is that no explicit assumption was made about the specific form of each M-pattern. Had we made such an assumption, we would have discovered the need for imposing a condition equivalent to (5-169) to ensure perfect replication in cases when each M-pattern consists of more than one element (aperture).

We mentioned earlier that the expressions on the left-hand side of equations (5-165) through (5-167) can be made as small as we wish by choosing J large enough. We now derive an expression for the noise in the replica resulting from the unwanted signals that occur when J is finite; we also derive the corresponding replica signal-to-noise ratio.

By analogy with (5-2)

$$R(x,y,t_k) = \sum_{r=\infty}^{\infty} \sum_{s=\infty}^{\infty} \mathcal{R}_{rs}(t_k)\, e^{i2\pi\left(\frac{rx}{2a''} + \frac{sy}{2b'}\right)} \tag{5-171}$$

where

$$\mathcal{R}_{rs}(t_k) = 4a''\, b'\, CT_j \sum_{n=-\infty}^{\infty} I_{rn}(t_{k-1} - \tau) \sum_{j=1}^{J} \mathcal{M}^*_{jrn}\, \mathcal{M}_{jrs}. \tag{5-172}$$

The desired portion of the replica is formed by the term for which

$$n = s$$

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The remaining terms produce unwanted signals which contribute to — what we shall call — a "confusion picture of the first kind". This picture $\mathcal{R}^{(1)}_{rs}(t_k)$ is given by

$$\mathcal{R}^{(1)}_{rs}(t_k) = 4a''b'CT_j \sum_{n=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau) \sum_{j=1}^{J} \mathcal{M}^*_{jrn}\,\mathcal{M}_{jrs}, \quad n \neq s \quad (5\text{-}173)$$

so that

$$\left\langle \left| \mathcal{R}^{(1)}_{rs}(t_k) \right|^2 \right\rangle =$$

$$\frac{(4a'b'CT)^2}{J^2} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau)\,\mathcal{I}^*_{rn'}(t_{k-1}-\tau) \left\langle \sum_{j=1}^{J} \sum_{j'=1}^{J} \mathcal{M}^*_{jrn}\mathcal{M}_{jrs}\mathcal{M}_{j'rn'}\mathcal{M}^*_{j'rs} \right\rangle$$

$$n \neq s \text{ and } n' \neq s. \quad (5\text{-}174)$$

Now consider

$$\left\langle \sum_{j=1}^{J} \sum_{j'=1}^{J} \mathcal{M}^*_{jrn}\,\mathcal{M}_{jrs}\,\mathcal{M}_{j'rn'}\mathcal{M}^*_{j'rs} \right\rangle$$

$$= \sum_{j=1}^{J} \sum_{j'=1}^{J} \left\langle \mathcal{M}^*_{jrn}\,\mathcal{M}_{jrs}\,\mathcal{M}_{j'rn'}\,\mathcal{M}^*_{j'rs} \right\rangle . \quad (5\text{-}175)$$

For all terms with j' different from j, $\mathcal{M}_{j'}$ and $\mathcal{M}_j$ are statistically independent. The same thing is true of their transforms,

Hence

$$\left\langle \mathcal{M}^*_{jrn}\mathcal{M}_{jrs}\mathcal{M}_{j'rn'}\,\mathcal{M}^*_{j'rs} \right\rangle$$

$$= \left\langle \mathcal{M}^*_{jrn}\mathcal{M}_{jrs} \right\rangle \left\langle \mathcal{M}_{j'rn'}\mathcal{M}^*_{j'rs} \right\rangle \quad (5\text{-}176)$$

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However

$$M^*_{jrn} M_{jrs} = P^*_{rn} P_{rs} \sum_{k=1}^{K} \sum_{k'=1}^{K} e^{i2\pi\left(\frac{r\xi_{jk}}{2a''} - \frac{r\xi_{jk'}}{2a''} + \frac{n\eta_{jk}}{2b'} - \frac{s\eta_{jk'}}{2b'}\right)} \quad (5\text{-}177)$$

and

$$\left\langle M^*_{jrn} M_{jrs} \right\rangle = P^*_{rn} P_{rs} \left\langle \sum_{k=1}^{K} \sum_{k'=1}^{K} e^{i2\pi\left(\frac{r\xi_{jk} - r\xi_{jk'}}{2a''} + \frac{n\eta_{jk} - s\eta_{jk'}}{2b'}\right)} \right\rangle . \quad (5\text{-}178)$$

Further

$$\left\langle \sum_{k=1}^{K} \sum_{k'=1}^{K} e^{i2\pi\left(\frac{r\xi_{jk} - r\xi_{jk'}}{2a''} + \frac{n\eta_{jk} - s\eta_{jk'}}{2b'}\right)} \right\rangle$$

$$\sum_{k=1}^{K} \sum_{k'=1}^{K} \left\langle e^{i2\pi\left(\frac{r\xi_{jk}}{2a''} - \frac{r\xi_{jk'}}{2a''} + \frac{n\eta_{jk'}}{2b'} - \frac{s\eta_{jk'}}{2b'}\right)} \right\rangle \quad (5\text{-}179)$$

Because of the statistical independence of ξ_{jk}, $\xi_{jk'}$, η_{jk}, and $\eta_{jk'}$, all terms on the right-hand side of (5-179) with k' different from k can be written as

$$\left\langle e^{i2\pi\frac{r\xi_{jk}}{2a''}} \right\rangle \left\langle e^{-i2\pi\frac{r\xi_{jk'}}{2a''}} \right\rangle \left\langle e^{i2\pi\frac{n\eta_{jk}}{2b'}} \right\rangle \left\langle e^{-i2\pi\frac{s\eta_{jk'}}{2b'}} \right\rangle . \quad (5\text{-}180)$$

This product vanishes, except when

$$r = n = s = o , \quad (5\text{-}181)$$

in which case it is equal to unity. Consequently, since we are considering only the case for which n is different from s, we can drop from (5-178) all of the terms for which k' is different from k. Then (5-178) becomes

$$\left\langle M^*_{jrn} M_{jrs} = P^*_{rn} P_{rs} \right\rangle \sum_{k=1}^{K} \left\langle e^{i2\pi\frac{n-s}{2b'}\xi_{jk}} \right\rangle , \quad (5\text{-}182)$$

which vanishes whenever n is different from s.

From (5-182), both sides of (5-176) vanish when

$$j' \neq j \text{ and } n \neq \text{ and/or } n' \neq s. \quad (5\text{-}183)$$

Accordingly, we can rewrite (5-174) as

$$\left\langle \left| R^{(1)}_{rs}(t_k) \right|^2 \right\rangle =$$

$$\frac{(4a''b'CT)^2}{J^2} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \mathcal{I}_{rn}(t_{k-1}-\tau)\mathcal{I}^*_{rn'}(t_{k-1}-\tau) \left\langle \sum_{J=1}^{J} \mathcal{M}^*_{jrn} \mathcal{M}_{jrs} \mathcal{M}_{jrn'} \mathcal{M}^*_{jrs} \right\rangle$$

$$n \neq s \qquad \text{and} \qquad n' \neq s \tag{5-184}$$

Again, consider

$$\left\langle \sum_{J=1}^{J} \mathcal{M}^*_{jrn} \mathcal{M}_{jrs} \mathcal{M}_{jrn'} \mathcal{M}^*_{jrs} \right\rangle$$

$$= \sum_{J=1}^{J} \left\langle \mathcal{M}^*_{jrn} \mathcal{M}_{jrs} \mathcal{M}_{jrn'} \mathcal{M}^*_{jrs} \right\rangle . \tag{5-184}$$

By an argument similar to that which lead to (5-182), this vanishes whenever n' is different from n. Using this result in (5-184), we have

$$\left| \mathcal{R}^{(1)}_{rs}(t_k) \right|^2 =$$

$$\frac{(4a''b'CT)^2}{J^2} \sum_{n=-\infty}^{\infty} \left| \mathcal{I}_{rn}(t_{k-1}-\tau) \right|^2 \left\langle \sum_{j=1}^{J} \left| \mathcal{M}_{jrn} \right|^2 \left| \mathcal{M}_{jrs} \right|^2 \right\rangle$$

$$n \neq s \tag{5-186}$$

We are not interested in the special case

$$\left\langle \left| \mathcal{R}^{(1)}_{oo}(t_k) \right|^2 \right\rangle$$

since this contribution is subtracted from the integrating screen. For all other cases,

$$\left\langle \left| M_{jrs} \right|^2 \right\rangle = \left| P_{rs} \right|^2 K \tag{5-187}$$

and we have

$$\left\langle \left| \mathcal{R}_{rs}(t_k) \right|^2 \right\rangle =$$

$$\frac{(4a''b'CT)^2 \left|\mathcal{P}_{rs}\right|^2 K}{J} \sum_{n=-\infty}^{\infty} \left|\mathcal{I}_{rn}(t_{k-1}-\tau)\right|^2 \left\langle \left|\mathcal{M}_{jrn}\right|^2 \right\rangle \tag{5-188a}$$

for

$$n \neq s \tag{5-188b}$$

For the general case, when r is different from zero

$$\left\langle \left|\mathcal{M}_{jrn}\right|^2 \right\rangle = \left|\mathcal{P}_{rn}\right|^2 K \tag{5-189}$$

and we have

$$\left\langle \left| \mathcal{R}^{(1)}_{rs}(t_k) \right|^2 \right\rangle =$$

$$\frac{(4a''b'CT)^2 \left|\mathcal{P}_{rs}\right|^2 \left|\mathcal{P}_{rn}\right|^2 K^2}{J} \sum_{n=-\infty}^{\infty} \left|\mathcal{I}_{rn}(t_{k-1}-\tau)\right|^2 =$$

$$\frac{(4a''b'CT)^2 \left|\mathcal{P}_{rs}\right|^2 K^2}{J} \left\{ \sum_{n=-\infty}^{\infty} \left|\mathcal{I}_{rn}(t_{k-1}-\tau)\right|^2 \left|\mathcal{P}_{rn}\right|^2 - \left|\mathcal{I}_{rs}(t_{k-1}-\tau)\right|^2 \left|\mathcal{P}_{rs}\right|^2 \right\} \tag{5-190}$$

For the special case where r is zero, we note that

$$\mathcal{M}^*_{joo}\mathcal{M}_{joo} = \left|\mathcal{P}_{oo}\right|^2 K^2 \tag{5-191}$$

so that

$$\left\langle \left| \mathcal{R}^{(1)}_{os}(t_k) \right|^2 \right\rangle =$$

$$\frac{(4a''b'CT)^2 \left|\mathcal{P}_{os}\right|^2 K^2}{J} \left\{ \begin{array}{l} \displaystyle\sum_{n=-\infty}^{\infty} \left|\mathcal{I}_{on}(t_{k-1}-\tau)\right|^2 \left|\mathcal{P}_{on}\right|^2 - \left|\mathcal{I}_{os}(t_{K-1}-\tau)\right|^2 \left|\mathcal{P}_{os}\right|^2 \\ \\ + (K^3 - K^2) \left|\mathcal{I}_{oo}(t_{k-1}-\tau)\right|^2 \left|\mathcal{P}_{oo}\right|^2 \end{array} \right\} \tag{5-192}$$

If, as is reasonable, we transmit only the ac component of the photosensor output and omit the dc component, this has the effect of removing $\mathcal{I}_{oo}(t_{k-1}-\tau)$, and (5-190) holds for all values of r.

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We now note that the space-variance of the confusion picture is given by

$$\sigma_{R_1}^2 = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left\langle \left| \mathcal{R}_{rs}^{(1)}(t_k) \right|^2 \right\rangle \tag{5-193a}$$

for

$$r \neq 0 \qquad \text{and/or} \qquad s \neq 0. \tag{5-193b}$$

Using (5-190), we have

$$\sigma_{R_1}^2 = \frac{(4a''b'CT)^2K^2}{J} \sum_{s=-\infty}^{\infty} \left\{ \sum_{r=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left| \mathcal{I}_{rn}(t_{k-1}-\tau) \right|^2 \left| \mathcal{P}_{rn} \right|^2 \left| \mathcal{P}_{rs} \right|^2 \right\}$$

$$- \frac{(4a''b'CT)^2K^2}{J} \left\{ \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left| \mathcal{I}_{rs}(t_{k-1}-\tau) \right|^2 \left| \mathcal{P}_{rs} \right|^4 \right\}, \tag{5-194}$$

Where, from (5-193), we omit the terms for which both r and s are zero. Further, since we have removed $\mathcal{I}_{oo}(t_{k-1}-\tau)$, we omit the terms for which both r and n are zero. We now suppose that condition (5-169) has been satisfied, so that for all terms of interest

$$\left| \mathcal{P}_{rs} \right|^2 = \frac{1}{4a''b'CTK} \tag{5-195}$$

Then we have from (5-194)

$$\sigma_{R_1}^2 = \frac{1}{J} \sum_{s=-\infty}^{\infty} \left\{ \sum_{r=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left| \mathcal{I}_{rn}(t_{k-1}-\tau) \right|^2 - \left| \mathcal{I}_{oo}(t_{k-1}-\tau) \right|^2 \right\}$$

$$- \frac{1}{J} \left\{ \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left| \mathcal{I}_{rs}(t_{k-1}-\tau) \right|^2 - \left| \mathcal{I}_{oo}(t_{k-1}-\tau) \right|^2 \right\}. \tag{5-196a}$$

for

$$s \neq 0. \tag{5-196b}$$

But

$$\sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left| \mathcal{I}_{rs}(t_{k-1}-\tau) \right|^2 - \left| \mathcal{I}_{oo}(t_{k-1}-\tau) \right|^2 \equiv \sigma_I^2, \tag{5-197}$$

the space-variance of the original image so that, from (5-196)

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$$\sigma^2_{R_1} = \frac{1}{J}\sum_{s=-\infty}^{\infty}\sigma^2_I - \frac{1}{J}\sigma^2_I \, . \tag{5-198a}$$

for

$$s \neq 0 \tag{5-198b}$$

In fact, we do not run over the full range of s in (5-198), but only between limits $\pm$ S, where S if defined by the relation

$$P_{rs} = 0 \qquad \text{for} \qquad |s| > S \geq N \tag{5-199}$$

where N/2b' is the highest spatial frequency of the original image in the y-direction. Hence, finally,

$$\sigma^2_{R_1} = \frac{2S-1}{J}\,\sigma^2_I \, . \tag{5-200}$$

It is reasonable to take the quantity $\sigma^2_{R_1}$ as a measure of the unwanted signals (or noise) corresponding to the confusion picture of the first kind. We shall refer to this quantity as "confusion noise of the first kind" and define the replica signal-to-noise ratio $(S/N)_{R_1}$ corresponding to this noise by

$$\left(\frac{S}{N}\right)_{R_1} \equiv \frac{\sigma^2_I}{\sigma^2_{R_1}} \tag{5-201}$$

Hence, from (5-201) and (5-202),

$$\left(\frac{S}{N}\right)_{R_1} = \frac{J}{2S-1} \qquad (5\text{-}202$$

The confusion picture of the first kind arises from a failure to meet the conditions (5-165) through (5-167). We shall refer to these conditions as the "orthogonality" condition, in addition to this confusion picture, we also get — what we shall call — a "confusion picture of the second kind" because we have satisfied conditions (5-169) and (5-170) only on the average. This picture occurs whenever

$$K \equiv \langle E_{jmn} E^*_{jmn} \rangle \neq E_{jmn} E^*_{jmn} \, . \tag{5-203}$$

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The corresponding noise — which we shall refer to as "confusion noise of the second kind" — can also be made as small as derived by choosing J large enough. We shall now prove this statement.

To this end, we start by considering the terms of the replica for which n is different from s. These are

$$\sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} 4a''b'CT_J \, \mathcal{I}_{rs}(t_{k-1}-\tau) \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} \, e^{i2\pi\left(\frac{rx}{2a''}+\frac{sy}{2b'}\right)} \tag{5-204}$$

Assuming that the functions $M_j(x,y)$ have been properly chosen so that

$$\left\langle \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} \right\rangle = \frac{1}{4a''b'CT_J} \, , \tag{5-205}$$

there remains a second confusion picture $R^{(2)}(x,y,t_k)$ which results — as mentioned earlier — from the fact that conditions (5-169) and (5-170) have been satisfied only on the average. We have

$$R^{(2)}(x,y,t_k) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} 4a''b'CT_J \mathcal{I}_{rs}(t_{k-1}-\tau) \left\{ \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} - \left\langle \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} \right\rangle \right\} e^{i2\pi\left(\frac{rx}{2a''}+\frac{sy}{2b'}\right)} \tag{5-206}$$

The expected value of $R^{(2)}(x,y,t_k)$ is zero at every point, since the expected value of every term on the right-hand side of (5-206) is zero because

$$\left\langle \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} - \left\langle \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} \right\rangle \right\rangle \equiv 0 . \tag{5-207}$$

Then, the confusion noise of the second kind $\sigma^2_{R_2}$ is given by

$$\sigma^2_{R_2} = (4a''b'CT_J)^2 \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left| \mathcal{I}_{rs}(t_{k-1}-\tau) \right|^2 \left\langle \left| \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} - \left\langle \sum_{j=1}^{J} \mathcal{M}^*_{jrs} \mathcal{M}_{jrs} \right\rangle \right|^2 \right\rangle \tag{5-208a}$$

$$r \neq 0 \qquad \text{and/or} \qquad s \neq 0. \tag{5-208b}$$

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We observe that

$$\sum_{j=1}^{J} M^*_{jrs} M_{jrs} = \sum_{j=1}^{J} \left| M_{jrs} \right|^2 ,$$

which is a real quantity. Then

$$\left\langle \left| \sum_{j=1}^{J} M^*_{jrs} M_{jrs} - \left\langle \sum_{j=1}^{J} M^*_{jrs} M_{jrs} \right\rangle \right|^2 \right\rangle$$

$$= \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 {}^2 - 2 \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \right\rangle + \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \right\rangle^2 \right\rangle$$

$$= \left\langle \left\{ \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \right\}^2 \right\rangle - \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \right\rangle^2 , \tag{5-209}$$

and we can write (5-208) as

$$\sigma^2_{R_2} = (4a''b'CT_J)^2 \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left| I_{rs}(t_{k-1}-\tau) \right|^2 \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 {}^2 \right\rangle - \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \right\rangle^2 \tag{5-210a}$$

for

$$r \neq 0 \qquad \text{and/or} \qquad s \neq 0. \tag{5-210b}$$

We now recall, from (5-141), that

$$M_{jrs} = P_{rs} E_{jrs} \tag{5-211}$$

so that

$$\sum_{j=1}^{J} \left| M_{jrs} \right|^2 = \left| P_{rs} \right|^2 \sum_{j=1}^{J} \left| E_{jrs} \right|^2 \tag{5-212}$$

and

$$\left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 {}^2 \right\rangle - \left\langle \sum_{j=1}^{J} \left| M_{jrs} \right|^2 \right\rangle^2$$

$$= \left| P_{rs} \right|^4 \left\{ \left\langle \sum_{j=1}^{J} \left| E_{jrs} \right|^2 {}^2 \right\rangle - \left\langle \sum_{j=1}^{J} \left| E_{jrs} \right|^2 \right\rangle^2 \right\} . \tag{5-213}$$

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In order to evaluate the second term on the right-hand side of (5-213), we write

$$\left\{\left\langle \sum_{j=1}^{J} \left|E_{jrs}\right|^2 \right\rangle\right\}^2 = \left\{\left\langle \sum_{j=1}^{J} \left|E_{jrs}\right|^2 \right\rangle\right\}^2 = \left\{\sum_{j=1}^{J} \left\langle \left|E_{jrs}\right|^2 \right\rangle\right\}^2 \tag{5-214}$$

we know, from (5-146) above, that

$$\left\langle \left|E_{jrs}\right|^2 \right\rangle = K \tag{5-215a}$$

for

$$r \neq 0 \qquad \text{and/or} \qquad s \neq 0. \tag{5-215b}$$

Hence,

$$\left\langle \left\{\sum_{j=1}^{J} \left|E_{jrs}\right|^2 \right\}^2 \right\rangle = \left\{ \sum_{j=1}^{J} K \right\}^2 = J^2 K^2. \tag{5-216}$$

In order to evaluate the first term, we write

$$\left\langle \sum_{j=1}^{J} \left|E_{jrs}\right|^2 {}^2 \right\rangle = \left\langle \sum_{j=1}^{J} \sum_{j'=1}^{J} \left|E_{jrs}\right|^2 \cdot \left|E_{j'rs}\right|^2 \right\rangle$$

$$= \sum_{j=1}^{J} \sum_{j'=1}^{J} \left\langle \left|E_{jrs}\right|^2 \cdot \left|E_{j'rs}\right|^2 \right\rangle . \tag{5-217}$$

For the terms, (J^2-J) in number, on the right-hand side of (5-117) for which j' is different from j, we recall that E_{jrs} and $E_{j'rs}$ are statistically independent of each other so that

$$\left\langle \left|E_{jrs}\right|^2 \cdot \left|E_{j'rs}\right|^2 \right\rangle = \left\langle \left|E_{jrs}\right|^2 \right\rangle \left\langle \left|E_{j'rs}\right|^2 \right\rangle = K^2. \tag{5-218}$$

Consequently (5-117) becomes

$$\left\langle \sum_{j=1}^{J} \left|E_{jrs}\right|^2 {}^2 \right\rangle = (J^2-J)K^2 + \sum_{j=1}^{J} \left\langle \left|E_{jrs}\right|^4 \right\rangle . \tag{5-219}$$

In order to evaluate $\left\langle \left|E_{jrs}\right|^4 \right\rangle$, we recall from (5-140) that

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$$E_{jrs} = \sum_{k=1}^{K} e^{-i2\pi\left(\frac{r\xi_{jk}}{2a''} + \frac{s\eta_{jk}}{2b'}\right)} \qquad (5\text{-}220)$$

so that

$$\left|E_{jrs}\right|^2 = E_{jrs}^{*}E_{jrs} = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} e^{i2\pi\left(\frac{r\xi_{jk_1}}{2a''} + \frac{s\eta_{jk_1}}{2b'}\right)} \cdot e^{-i2\pi\left(\frac{r\xi_{jk_2}}{2a''} + \frac{s\eta_{jk_2}}{2b'}\right)} . \qquad (5\text{-}221)$$

Therefore $\left|E_{jrs}\right|^4$ can be written as a quadruple sum over indices k_1, k_2, k_3, and k_4 with the sign of the exponent positive for k_1 and k_3 and negative for k_2 and k_4. When we consider the K^4 terms in the quadruple sum, we find the following:

(a) There are K terms in which all four subscripts (k_1, k_2, k_3, and k_4) have the same value. The expected value of each of these terms is 1.

(b) There are 4K(K-1) terms in which three subscripts have the same value and the fourth has a different value. The expected value of each of these terms is zero.

(c) There are K(K-1) terms for which k_1 and k_2 have one value and k_3 and k_4 have another value. The expected value of each of these terms is 1.

(d) There are K(K-1) terms for which k_1 and k_3 have one value and k_2 and k_4 have another value. The expected value of each of these terms is zero.

(e) There are K(K-1) terms for which k_1 and k_4 have one value and k_2 and k_3 have another value. The expected value of each of these terms is one.

(f) There are K(K-1)(K-2) terms in which three of the subscripts have different values and the fourth subscript has a value equal to one of the other three. The expected value of each of these terms is zero.

(g) There are K(K-1)(K-2)(K-3) terms in which all four subscripts have different values. The expected value of each of these terms is zero.

Thus, in examining the expected value of the quadruple sum which is equal to $\left|E_{jrs}\right|^4$, we have

$(2k^2-k)$ terms with a value of 1

(k^4-2k^2+k) terms with a value of 0

and

$$\left\langle \left|E_{jrs}\right|^4 \right\rangle = 2K^2-K. \quad (5\text{-}222)$$

Using (5-222) in (5-219), we have

$$\left\langle \left(\sum_{j=1}^{J} \left|E_{jrs}\right|^2\right)^2 \right\rangle = (J^2-J)K^2 + J(2K^2-K) = J^2K^2 + JK^2 - JK. \quad (5\text{-}223)$$

Substituting (5-216) and (5-223) in (5-213), we have

$$\left\langle \left(\sum_{j=1}^{J} \left|M_{jrs}\right|^2\right)^2 \right\rangle - \left\langle \sum_{j=1}^{J} \left|M_{jrs}\right|^2 \right\rangle^2 = \left|P_{rs}\right|^4 J(K^2-K). \quad (5\text{-}224)$$

Substituting, in turn, this expression into (5-210) we have

$$\sigma_{R_1}^2 = (4a''b'CT_J)^2 \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left|I_{rs}(t_{k-1}-\tau)\right|^2 \left|P_{rs}\right|^4 J(K^2-K) \quad (5\text{-}225)$$

From (5-170), we recall that P(x,y) was chosen so that

$$4a''b'CT_J \left|P_{rs}\right|^2 JK^2 = 1 \quad (5\text{-}226a)$$

$$r \neq o \qquad \text{and/or} \qquad s \neq 0 \quad (5\text{-}226b)$$

over the region of interest. Substituting this value in (5-225), we have for the confusion noise of the second kind

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$$\sigma_{R_2}^2 = \frac{J(K^2-K)}{J^2K^2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left| \Gamma_{rn}(t_{k-1}-\tau) \right|^2 = \frac{1-\frac{1}{K}}{J} \sigma_I^2 , \qquad (5\text{-}227)$$

and for the corresponding signal-to-noise ratio $(S/N)_{R_2}$

$$\left(\frac{S}{N}\right)_{R_2} \equiv \frac{\sigma_I^2}{\sigma_{R_2}} = \frac{1-\frac{1}{K}}{J} . \qquad (5\text{-}228)$$

The replica signal-to-noise ration $(S/N)_R$ corresponding to both kinds of confusion noise

$$\left(\frac{S}{N}\right)_R = \frac{\sigma_I^2}{\sigma_{R_1}^2 + \sigma_{R_2}^2} = \frac{1}{\frac{2S-1}{J} + \frac{1-\frac{1}{K}}{J}} = \frac{J}{2S-\frac{1}{K}} \qquad (5\text{-}230)$$

Therefore, for large values of K, we have

$$\left(\frac{S}{N}\right)_R \approx \frac{J}{2S} . \qquad (5\text{-}231)$$

The total displacement of the transmissivity patterns during the integration interval T is

$$vT = 2a''J. \qquad (5\text{-}64)$$

From(5-230) and (5-64), it follows that

$$\left(\frac{S}{N}\right)_R = \frac{vT}{2a''(2S-\frac{1}{K})} \qquad (5\text{-}232)$$

and from(5-31) and (5- 64) that

$$\frac{S}{N}_R \approx \frac{vT}{4a''S} \qquad (5\text{-}233)$$

for K large.

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5.8 Alternative Form of Physically Realizable Weighting Functions Satisfying the Second Set of Less Restrictive Conditions.

In 5.5, above, we considered the case of a transmissivity pattern having internal periodicity, and showed that such periodicity (when properly chosen) automatically satisfied the orthogonality conditions. The particular form we choose for W(x,y) in 5.6 was

$$W(x,y) = \sum_{j=1}^{J} M\left(x - j\frac{2a}{J},\ y - j\frac{2b'}{J}\right) \tag{5-123}$$

where

$$\frac{2a}{J} = 2a' . \tag{5-234}$$

The equivalent of condition (5-122 turned out to be simply

$$\left| \mathcal{M}_{r,-r} \right|^2 = \frac{1}{4ab'CTJ} \tag{5-134}$$

Now, as in 5.7, we let

$$M(x,y) = \sum_{k=1}^{K} P_k(x,y) = \sum_{k=1}^{K} P(x-\xi_k,\ y-\eta_k) \tag{5-235}$$

where ξ_k and η_k are a set of statistically independent quantities, randomly distributed over the intervals from -a to +a and from -b' to +b' respectively, having uniform density with respect to k. Following the same line of reasoning as was used in 5.7 (cf 5-139),

$$\mathcal{M}_{mn} = \mathcal{P}_{mn} \sum_{k=1}^{K} e^{-i2\pi\left(\frac{m\xi_k}{2a} + \frac{n\eta_k}{2b'}\right)} \tag{5-236}$$

whence, as before, (cf 5-148)

$$\left| \mathcal{M}_{oo} \right|^2 = \left| \mathcal{P}_{oo} \right|^2 K^2 \tag{5-237}$$

and (cf 5-149)

$$\left\langle \left| \mathcal{M}'_{mn} \right|^2 \right\rangle = \left| \mathcal{P}_{mn} \right|^2 K \tag{5-238a}$$

for

$$m \neq o \qquad \text{and/or} \qquad n \neq o . \tag{5-238b}$$

From (5-134) and (5-238), we obtain for

$$\left| P_{r,-r} \right|^2 = \frac{1}{4ab' CTKJ} \tag{5-239a}$$

for

$$r \neq o. \tag{5-239b}$$

As in the earlier case, we observe that we must subtract the dc term which arises for r equal to zero. As before, we can remove this dc term by subtracting it at the integrating screen. J is now determined, as in the non-random cases, by the highest spatial frequencies, in the y-direction of the original image and the transmissivity pattern.

Confusion noise of the first kind —which results from not satisfying the orthogonality conditions — does not appear in this case. However, confusion noise of the second kind — which in this case results from randomly displaced versions of the desired replica —does appear when the length vT of the transmissivity patterns during the integration interval T is finite. We shall now derive an expression for this noise.

The replica contains no "non-orthogonal" terms in this case. We have for $R(x,y,t_k)$ only

$$R(x,y,t_k) = \sum_{r=-\infty}^{\infty} R_{r,-r}(t_k)\, e^{i2\pi\left(\frac{rx}{2a} - \frac{sy}{2b'}\right)}, \tag{5-240}$$

where

$$R_{r,-r}(t_k) = 4ab'CTJ\, I_{r,-r}(t_{k-1}-\tau) \left| M_{r,-r} \right|^2 \tag{5-241}$$

Drawing on 5.7, we can write

$$\sigma_{R_1}^2 \equiv o \tag{5-242}$$

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and

$$\sigma^2_{R_2} = (4ab'CTJ)^2 \sum_{r=-\infty}^{\infty} \left| \mathcal{I}_{r,-r}(t_{k-1}-\tau) \right|^2 \left\{ \left\langle \left| \mathcal{M}_{r,-r} \right|^4 \right\rangle - \left\langle \left| \mathcal{M}_{r,-r} \right|^2 \right\rangle^2 \right\} \tag{5-243a}$$

for

$$r \neq o. \tag{5-243b}$$

As before,

$$\mathcal{M}_{r,-r} = \mathcal{P}_{r,-r} \; E_{r,-r} , \tag{5-244}$$

where

$$E_{r,-r} = \sum_{k=1}^{K} e^{-i2\pi\left(\frac{r\xi_k}{2a} - \frac{r\eta_k}{2b'}\right)} \tag{5-245}$$

so that

$$\left\langle \left| \mathcal{M}_{r,-r} \right|^4 \right\rangle = \left| \mathcal{P}_{r,-r} \right|^4 \left\langle \left| E_{r,-r} \right|^4 \right\rangle \tag{5-246}$$

and

$$\left\langle \left| \mathcal{M}_{r,-r} \right|^2 \right\rangle^2 = \left| \mathcal{P}_{r,-r} \right|^4 \left\langle \left| E_{r,-r} \right|^2 \right\rangle^2 . \tag{5-247}$$

Then

$$\left\langle \left| E_{r,-r} \right|^4 \right\rangle = 2K^2 - K \tag{5-248}$$

and

$$\left\langle \left| E_{r,-r} \right|^2 \right\rangle^2 = K^2. \tag{5-249}$$

Accordingly,

$$\sigma^2_{R_2} = (4ab'CTJ)^2 \sum_{r=-\infty}^{\infty} \left| \mathcal{I}_{r,-r}(t_{k-1}-\tau) \right|^2 \left| \mathcal{P}_{r,-r} \right|^4 (K^2-K). \tag{5-250}$$

However, over the region of interest,

$$4ab'CTJ \left| \mathcal{P}_{r,\ r} \right|^2 K=1 \tag{5-251}$$

so that we have

$$\sigma^2_{R_2} = \left(1 - \frac{1}{K}\right) \sum_{r=-\infty}^{\infty} \left| \mathcal{I}_{r,-r}(t_{k-1}-\tau) \right|^2 = \left(1 - \frac{1}{K}\right) \sigma^2_I . \tag{5-252}$$

To reduce the confusion noise of the second kind, we use a transmissivity pattern of m discrete sections, each of length $2a'J$ as described in 5.8. The interval of expansion for $\mathcal{M}_{mn}$ and $\mathcal{P}_{mn}$ is now $2a'J$ by $2b'$. As in 5.7, we separate each transmissivity pattern of length $2a'J$ by opaque regions of length $2a'$. Then integrating over a time T, given by

$$vT = 2a = m(J+1)\ 2a', \tag{5-253}$$

we obtain for the total confusion noise σ_R,

$$\sigma_R = \sigma_{R_2}^2 = \frac{1-\frac{1}{K}}{m} \quad \sigma_I^2 \tag{5-254}$$

and correspondingly

$$(\frac{S}{N})_R = (\frac{S}{N})_{R_2} = \frac{m}{1-\frac{1}{K}} \approx m. \tag{5-255}$$

Using this value of m in (5-253), we obtain for the total displacement vT of the transmissivity patterns interval T during the integration

$$vT \approx (\frac{S}{N})_R \ (J+1)\ 2a' \approx (\frac{S}{N})_R\ 2a'J \tag{5-256}$$

or

$$(\frac{S}{N})_R \approx \frac{VT}{2a'J} \tag{5-257}$$

for large values of K and J. From earlier considerations, we require that

$$J > N+S, \tag{5-258}$$

where N is defined by the relation

$$\mathcal{I}_{mn}(t_{k-1}-\tau) \equiv o \qquad \text{for} \qquad |n| > N \tag{5-259a}$$

$$P_{rs} \equiv o \qquad \text{for} \qquad |s| > S. \tag{5-259b}$$

Using this value of J in (5-257), we obtain

$$(\frac{S}{N})_R = \frac{VT}{2a'(N+S)}. \tag{5-260}$$

We note that the same value of $(S/N)_R$ can be achieved with a shorter

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displacement vT when random transmissivity patterns with internal periodicity are used instead of random transmissivity patterns without internal periodicity since

$$a' < a'', \tag{5-261}$$

and S is chosen so that

$$S \geq N. \tag{5-262}$$

5.9 Discussion of the General Forms of Weighting Functions Providing Perfect Replicas

We now review and discuss the results obtained so far in this section.

We first derived a set of sufficient conditions for the, in principle, perfect replication of bandlimited images produced by weighting functions implemented with moving time-invariant transmissivity patterns (scanning aperture patterns). We discovered that no weighting functions of this type satisfy these conditions — even if these functions consist of transmissivity patterns of infinite length.

We next noticed that the sufficient conditions mentioned above were not necessary ones, and sought to find alternative less restrictive conditions. We found two such sets of conditions, and discovered two general forms of non-random transmissivity patterns (see Subsections 5.4 and 5.6) that satisfy the first set and the second set of less restrictive conditions, respectively. We also discovered (see Subsections 5.7 and 5.8) a set of two general forms of random transmissivity patterns that can also be made — albeit on a statistical basis — to satisfy the first and second set of less restrictive conditions, respectively.

We note that conventionally instrumented single-spot line-scan television is a special case of the general forms discussed in Subsections (5.4) and (5.6). In effect both these forms reduce to single-spot line-scan television when the M-patterns discussed in these two subsections consist of a single small hole. (Note that the length d_e of the opaque regions separating transparent regions is, for all practical purposes, zero in this case.)

Instead of using M-patterns consisting of a single hole, we can use M-patterns consisting of a cluster of holes, possibly of different sizes and shapes, to modify the frequency response of these patterns. (In this case the M-patterns of Subsection (5.4) — but not those of Subsection 5.7 — must be separated by opaque regions of non-zero length d_e.) We have not investigated the properties of such patterns.

An alternative way of modifying the frequency response of the M-patterns is to shade the transmissivity of each transparent element appropriately. "Clear holes" and two types of shaded elements are discussed in Subsection 6.2.

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We observe that the orthogonality condition (see 5-75) ensures that no unwanted "spectral components" appear in the replica, that is that the replica contains only spectral components which belong to the original image. We saw that this condition can always be satisfied in principle. However, it may not always be satisfied in practice. When it is not, "confusion noise of the first kind" occurs which is similar in nature to that discussed in the case of purely random transmissivity patterns.

We also observe that the second condition (see 5-76) ensures that all the spectral components contained in the original image are reproduced with the correct amplitude and phase. That is, the second condition ensures that the frequency response of a pair of transmissivity patterns is flat (and free from phase shift) over the spatial-frequency band containing all spectral components with significant intensity; and that this frequency response is properly normalized. This latter characteristic of the second condition is trivial and we shall henceforth refer to the second condition as the (spatial-frequency) flatness condition.[1]

We conclude these remarks on spot-scan television by noting that the second set of less restrictive conditions consists of only a single condition; the flatness condition. The orthogonality condition does not appear because it is automatically satisfied by choosing the internal periodicity of the transmissivity patterns appropriately.

We next considered two general forms of transmissivity patterns for area-scan television.

The first form of random transmissivity patterns is made up of M-patterns of length 2a'' (which normally is equal to 4a') which in turn consist of randomly placed P-patterns. These latter patterns could consist of a cluster of transmissivity elements; however, we believe they will usually be chosen to consist of a single element. This type of transmissivity pattern produces confusion noise of the first and second kind. To decrease this noise to an acceptable level, we have to use successively a number of different M-patterns

[1]In practice perfect flatness over the band of the original image is seldom required, and may even be undesirable, as discussed in Subsection 6.3 in connection with equations (6-33) and (6-34).

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during the integration interval T.

The second form of random transmissivity patterns is made up of periodically-related M-patterns in such a way that they form a single W-pattern with an internal periodicity in a direction which makes, in general, a small angle with the x-direction. This pattern produces no confusion noise of the first kind but does produce confusion noise of the second kind which can be decreased by using successively a number of W-patterns during the integration interval. However, the transmissivity-pattern length vT required in this case to obtain a given replica signal-to-noise ratio is less than that required for purely random patterns.

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6. SPECIFIC FORMS OF WEIGHTING-FUNCTION ELEMENTS (TRANSMISSIVITY-PATTERN ELEMENTS) AND QUALITY OF THE RESULTING REPLICAS

So far we have been concerned only with the general forms of weighting functions — and their associated transmissivity patterns — that can in principle provide perfect replicas of bandlimited images. We shall now consider some specific forms of weighting-function elements and the quality of the replicas obtainable with these elements. We shall first, however, discuss the "quality" function. This function is usefull in assessing the quality of replicas obtained with different transmissivity-pattern elements.

6.1 The Quality Function

We substitute the expression for $R_u(t)$ given by (5-16) into (2-4) and obtain

$$R(x,y,t_k) = C_R C_{TR} C_T \int_{t_{k-1}}^{t_k} \left\{ \int_{-a'}^{a'} \int_{-b'}^{b'} I(x',y',t-\tau) W_I(x',y',t-\tau) dx'dy' \right\} W_R(x,y,t)dt.$$

This expression may be re-written as

$$R(x,y,t) = \int_{-a'}^{a'} \int_{-b'}^{b'} I(x',y',t-\tau) \left\{ C \int_{t_{k-1}}^{t_k} W_I(x',y',t-\tau) W_R(x,y,t)dt \right\} dx'dy' \qquad (6\text{-}1)$$

by changing the order of integration. By Assumption D, the functions W_I and W_R are periodic with respect to time, having a common period

$$T \equiv t_k - t_{k-1}. \qquad (2\text{-}5)$$

The integral in brackets then depends only on the four spatial coordinates x, y, x', and y'. We can therefore represent it by a function $Q(x,y,x',y')$, so that

$$R(x,y,t_k) = \int_{-a'}^{a'} \int_{-b'}^{b'} I(x',y',t_{k-1}-\tau)\, Q(x,y,x',y')\, dx'dy'. \qquad (6\text{-}2)$$

Now let α and β be two positive quantities such that

$$\alpha \geq a' \text{ and } \beta \geq b'. \qquad (6\text{-}3)$$

We can now write

$$I(x',y',t_{k-1}-\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t_{k-1}-\tau)\, e^{i2\pi\left(\frac{mx'}{2\alpha} + \frac{ny'}{2\beta}\right)}, \qquad (6\text{-}4)$$

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where

$$\mathcal{I}_{mn}(t_{k-1}-\tau) = \frac{1}{4\alpha\beta} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} I(x',y',t_{k-1}-\tau)\, e^{-i2\pi\left(\frac{mx'}{2\alpha}+\frac{ny'}{2\beta}\right)} dx'\,dy'. \qquad (6\text{-}5)$$

Then, substituting $I(x'y't_{k-1}-\tau)$ from (6-4) into (6-2), we obtain

$$R(x,y,t_k) = \frac{1}{4\alpha\beta} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t_{k-1}-\tau) \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} Q(x,y,x',y')\, e^{i2\pi\left(\frac{mx'}{2\alpha}+\frac{ny'}{2\beta}\right)} dx'dy'. \qquad (6\text{-}6)$$

We can also write

$$R(x,y,t_k) = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} \mathcal{R}_{rs}(t_k) e^{i2\pi\left(\frac{rx}{2\alpha}+\frac{sy}{2\beta}\right)} \qquad (6\text{-}7)$$

where

$$\mathcal{R}_{rs}(t_k) = \frac{1}{4\alpha\beta}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} R(x,y,t_k) e^{-i2\pi\left(\frac{rx}{2\alpha}+\frac{sy}{2\beta}\right)} dx\,dy \qquad (6\text{-}8)$$

and

$$\mathcal{R}_{rs}(t_k) \equiv \frac{1}{4\alpha\beta}\int_{-\alpha}^{-\alpha}\int_{-\beta}^{-\beta} R(x,y,t_k)\, e^{-i2\pi\left(\frac{rx}{2\alpha}+\frac{sy}{2\beta}\right)} dx\,dy$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathcal{I}_{mn}(t_{k-1}-\tau)\cdot\frac{1}{16\alpha^2\beta^2}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} Q(x,y,x',y')\, e^{i2\pi\left(\frac{mx'}{2\alpha}+\frac{ny'}{2\beta}-\frac{rx}{2\alpha}-\frac{sy}{2\beta}\right)} dx'dy'\,dx\,dy. \qquad (6\text{-}9)$$

The integral

$$\frac{1}{16\alpha^2\beta^2}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} Q(x,y,x',y')\, e^{i2\pi\left(\frac{mx'}{2\alpha}+\frac{ny'}{2\beta}-\frac{rx}{2\alpha}-\frac{sy}{2\beta}\right)} dx'dy'dx\,dy$$

$$=\frac{1}{16\alpha^2\beta^2}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} Q(x,y,-x',-y') e^{-i2\pi\left(\frac{rx}{2\alpha}+\frac{sy}{2\beta}+\frac{mx'}{2\alpha}+\frac{ny'}{2\beta}\right)} dx\,dy\,dx'dy'$$

$$\equiv \tilde{Q}_{rsmn}. \qquad (6\text{-}10)$$

This is the four-dimensional Fourier transform of $Q(x,y,-x'-y')$ so that

$$Q(x,y,-x',-y') = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \tilde{Q}_{rsmn} e^{i2\pi\left(\frac{rx}{2\alpha}+\frac{sy}{2\beta}+\frac{mx'}{2\alpha}+\frac{ny'}{2\beta}\right)} \qquad (6\text{-}11)$$

and

$$Q(x,y,x',y',) = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\widetilde{Q}_{rsmn} e^{i2\pi\left(\frac{rx}{2\alpha}+\frac{sy}{2\beta}-\frac{mx'}{2\alpha}-\frac{ny'}{2\beta}\right)} \tag{6-12}$$

From (6-9) and (6-10), we obtain

$$\mathcal{R}_{rs}(t_k) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathcal{I}_{mn}(t-\tau)\widetilde{Q}_{rsmn}. \tag{6-13}$$

In the special case where

$$\widetilde{Q}_{rsmn} = 0 \qquad \text{for} \qquad m \neq r \qquad \text{and/or} \qquad n \neq s, \tag{6-14}$$

which always happens when we have satisfied the orthogonality conditions, we have

$$Q(x,y,x',y') = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\widetilde{Q}_{rsrs} e^{i2\pi\left(\frac{r(x-x')}{2\alpha}+\frac{s(y-y')}{2\beta}\right)} \tag{6-15}$$

so that Q is a function only of the two differences $(x-x')$ and $(y-y')$. From (6-13) and (6-14), it follows that

$$\widetilde{Q}_{rsrs} = \frac{\mathcal{R}_{rs}(t_k)}{\mathcal{I}_{rs}(t_{k-1}-\tau)}, \tag{6-16}$$

and consequently

$$Q(x-x',y-y') = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\frac{\mathcal{R}_{rs}(t_k)}{\mathcal{I}_{rs}(t_{k-1}-\tau)} e^{i2\pi\left(\frac{r(x-x')}{2\alpha}+\frac{s(y-y')}{2\beta}\right)}, \tag{6-17}$$

where 2α and 2β are chosen, of course, to match the basic intervals of expansion originally used in defining $\mathcal{R}_{rs}$ and $\mathcal{I}_{rs}$.

For the final set of less restrictive conditions,

$$Q(x-x', y-y') = 4a''b'CT\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\mathcal{M}^{*}_{rs}\mathcal{M}_{rs} e^{i2\pi\left(\frac{r(x-x')}{2a''}+\frac{s(y-y')}{2b'}\right)} \tag{6-18}$$

or

$$Q(\xi,\eta) = 4a''b'CT \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} M^*_{rs} M_{rs} e^{i2\pi\left(\frac{r\xi}{2a''} + \frac{s\eta}{2b'}\right)} \tag{6-19}$$

Since

$$\frac{1}{4a''b'} \int_{-a''}^{a''} \int_{-b'}^{b'} M(x,y)M(x+\xi, y+\eta)dxdy$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} M_{rs} e^{i2\pi\left(\frac{r\xi}{2a''} + \frac{s\eta}{2b'}\right)} \frac{1}{4a''b'} \int_{-a''}^{a''} \int_{-b'}^{b'} M(x,y) e^{i2\pi\left(\frac{rx}{2a''} + \frac{sy}{2b'}\right)} dxdy \tag{6-20}$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} M_{rs} e^{i2\pi\left(\frac{r\xi}{2a''} + \frac{sy}{2b'}\right)} M^*_{rs} , \tag{6-21}$$

$$Q(\xi, \eta) = CT \int_{-a''}^{a''} \int_{-b'}^{b'} M(x,y)M(x+\xi, y+\eta)dxdy \tag{6-22}$$

For the second set of less restrictive conditions, we have — omitting the intermediate steps —

$$Q(\xi, \eta) = CT \int_{-a}^{a} \int_{-b'}^{b'} W(x,y)W(x+\xi, y+\eta)dxdy. \tag{6-23}$$

$$= CTJ \int_{-a'}^{a'} \int_{-b'}^{b'} M(x,y)M(x+\xi, y+\eta)dxdy. \tag{6-24}$$

For the purely random weighting functions of Section 5.7, Q is defined only in a statistical sense as

$$\langle Q(\xi, \eta)\rangle = CTK \int_{-a''}^{a''} \int_{-b'}^{b'} P(x,y)P(x+\xi, y+\eta)dxdy \tag{6-25}$$

For the quasi-random weighting functions of Section 5.8, we have similarly

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$$\langle Q(\xi, \eta) \rangle = CTKJ \int_{-a''}^{a''} \int_{-b'}^{b'} P(x,y)P(x+\xi, y+\eta)\,dx\,dy. \qquad (6\text{-}26)$$

6.2 Quality Functions and Space-Frequency Functions of Three Different Transmissivity-Pattern Elements (Apertures)

In the case of a rectangular hole, of width Δy, and height Δy, the quality function is given by

$$\frac{Q(\xi,\eta)}{Q(o,o)} = \frac{(\Delta x - |\xi|)(\Delta y - |\eta|)}{\Delta x \Delta y} \quad \text{for} \quad \begin{cases} |\xi| \leq \Delta x \\ \text{and} \\ |\eta| \leq \Delta y \end{cases} \tag{6-27a}$$

$$= 0 \quad \text{for} \quad \begin{cases} |\xi| > \Delta x \\ |\eta| > \Delta y \end{cases} \tag{6-27b}$$

The space-frequency (response) function is given by the transform of $Q(\xi,\eta)$ and is, after normalization,

$$\frac{\tilde{Q}(fx, fy)}{\tilde{Q}(o,o)} = \frac{\mathrm{Sin}^2 \pi fx \Delta x \mathrm{Sin}^2 \pi fy \Delta y}{(\pi^2 fxfy \Delta x \Delta y)^2} \tag{6-28}$$

For a Gaussian aperture defined by a transmissivity function $M(x,y)$ or $P(x,y)$ having the form

$$M(x,y) = P(x,y) = e^{-\left(\frac{x^2}{\Delta x^2} + \frac{y^2}{\Delta y^2}\right)}, \tag{6-29}$$

the normalized quality function is

$$\frac{Q(\xi,\eta)}{Q(o,o)} = e^{-\frac{1}{2}\left(\frac{\xi^2}{\Delta x^2} + \frac{\eta^2}{\Delta y^2}\right)} \tag{6-30}$$

and, the normalized space-frequency function is

$$\frac{\tilde{Q}(fx, fy)}{\tilde{Q}(o,o)} = e^{-2\pi^2(f_x{}^2 \Delta x^2 + f_y{}^2 \Delta y^2)}. \tag{6-31}$$

For a transmissivity function

$$M(x,y) = P(x,y) = \frac{\sin 2\pi\frac{x}{\Delta x}\ \sin 2\pi\frac{y}{\Delta y}}{4\pi^2\frac{xy}{\Delta x\Delta y}}, \tag{6-32}$$

the normalized quality function is

$$\frac{Q(\xi,\eta)}{Q(o,o)} = \frac{\sin 2\pi\,\xi/\Delta x\,\sin 2\pi\,\eta/\Delta y}{4\pi^2\frac{\xi\eta}{\Delta x\Delta y}} \tag{6-33}$$

and the normalized space-frequency function is

$$\frac{\widetilde{Q}(fxfy)}{\widetilde{Q}(o,o)} = 1 \qquad \text{for} \quad \begin{cases} |fx| \le \frac{1}{\Delta x} \\ \text{and} \\ |fy| \le \frac{1}{\Delta y} \end{cases} \tag{6-34a}$$

$$= 0 \qquad \text{for} \quad \begin{cases} |fx| > \frac{1}{\Delta x} \\ |fy| > \frac{1}{\Delta y} \end{cases} \tag{6-34b}$$

We cannot, of course, instrument a negative transmissivity function — as implied by (6-32). We can, however, achieve the same effect by subtracting the appropriate uniform value from R(x,y,t) on the integrating screen on which the replica is displayed.

6.3 Resolution

We have seen that no confusion noise exists in the case of a spot-scan system when the orthogonality condition is satisfied. Hence in this case the fineness of detail of the original image which is reproduced in the replica depends entirely on the quality function. However, more scanning lines are required to satisfy the orthogonality conditions when the quality function is rectangular than when it has the form sinx/x. In effect, in theory, an infinite number of such lines is required in the former case and a number of lines equal to 2N in the latter, where N is such that

$$I_{mn} = 0 \qquad \text{for} \qquad n > N. \tag{6-35}$$

(In practice of course only a finite number of spectral lines with significant intensity exists in the spatial-frequency response, and hence the number of scanning lines required to avoid detectable confusion noise is finite).

We now define the resolution of the replica to be the fineness of its detail in the absence of any noise and, in particular, of confusion noise. The quality function, or the spatial-frequency function, clearly provides an excellent description of this resolution. We note that the resolution provided by a quality function of the form sinx/x is much coarser than that provided by a "rectangular" quality function when both of these functions refer to apertures of the same size. The "gaussian" quality function represents a useful compromise between the conflicting requirements of fine resolution and the absence of confusion noise.

We now discuss the resolution provided by the normalized gaussian quality function (6-30) and its corresponding transform (6-31) when the dimensions of the aperture are

$$\Delta x = \Delta y = \Delta. \tag{6-36}$$

The transmissivity of this aperture drops (in any direction from the center) to 1/e of its maximum value at the center of the aperture. The normalized quality and spatial-frequency functions drop to 1/e at a distance $\Delta\sqrt{2}$ and $1/\Delta\pi\sqrt{2}$ from the center, respectively. If one thinks in terms of a square aperture with sides equal to Δ, the first null of the spatial-frequency (response) function occurs at $1/\Delta$ in either the x or y-direction. This function is closely approximated by the circular gaussian aperture defined by

$$e^{-\frac{x^2 + y^2}{(1425\Delta)^2}} , \tag{6-37}$$

which has an "effective" diameter equal to the cut-off wavelength of the square aperture. That is, loosely speaking, the diameter of the gaussian aperture is equal to the reciprocal of the cut-off frequency. We note that the quality and spatial-frequency functions of the circular gaussian aperture are uniform in all directions whereas those of the square gaussian aperture are not.

We conclude this discussion on resolution by observing that, in the absence of confusion noise, the quality function depends only on the coordinate differences $(x-x')$ and $(y-y')$ and not on the coordinates x and y themselves (see 6-17). Hence the resolution of the replica is spatially uniform.

6.4 Contrast

We define the contrast c_R of the replica by

$$c_R = \frac{\overline{R^2}(x,y,t_k)}{\overline{R}^2(x,y,t_k)} , \tag{6-38}$$

where the bar indicates a space average. This quantity can be adjusted as desired by subtracting an appropriate amount of "light" at the integrating screen.

6.5 Relative Total Intensity Range

We define the relative total intensity range i_R of the replica by

$$i_R \equiv \frac{R_{max} - R_{min}}{\overline{R}} . \tag{6-39}$$

This quantity can also be adjusted as desired by subtracting an appropriate amount of light at the integrating screen.

6.6 Linearity with Respect to Intensity and Spatial Distortion

The operations performed by the proposed instrumentation — whether they be used for a spot-scan or area-scan system — are linear and should introduce even in actual practice no significant non-linearity or spatial distortion.

7. RELATIVE SYSTEM PERFORMANCE OF AREA-SCAN AND SPOT-SCAN TELEVISION

7.1 Spot-Scan System

Let[1]

$$f'_{x_{max}} = \frac{1}{\Delta x'} \qquad \text{and} \qquad f'_{y_{max}} = \frac{1}{\Delta y'} \qquad (7\text{-}1a),(7.1b)$$

be the highest significant space frequencies in the original image corresponding to the x' and the y'-directios, respectively. Then the total number of independent samples N'_1 in an original image with dimensions 2a' by 2b' is

$$N'_1 = 2(2a')\frac{1}{\Delta x'} \cdot 2(2b')\frac{1}{\Delta y'} = 16a'b'\frac{1}{\Delta x'}\frac{1}{\Delta y'} . \qquad (7\text{-}2)$$

The corresponding highest time-frequency f and time-bandwidth B are

$$f' = B' = vf'_x = \frac{v}{\Delta x'} , \qquad (7\text{-}3)$$

where v is the speed of the transmissivity patterns. This relation is independent of the form of the transmissivity patterns in the absence of both types of confusion noise and of effective photosensor and receiver noise. It provides in this ideal case an expression for the minimum bandwidth required to obtain a perfect replica.

In practice the value of the bandwidth B, given by expression (7-3), must be increased — even in the case of a spot-scan system — if a high-fidelity replica is to be obtained. In particular, the time-bandwidth B must have a value larger than the one given by (7-3) if confusion is to be avoided with a spot-scan system when — as is usually the case — the space-filter function of the scanning aperture (transmissivity-pattern element) has a wider bandwidth than f'. In fact, if the space-frequency response of the scanning aperture used has maximum significant frequencies

$$f_{x_{max}} = \frac{1}{\Delta x} \qquad \text{and} \qquad f_{y_{max}} = \frac{1}{\Delta y} ,$$

[1] Here $\Delta x'$ and $\Delta y'$ are the "widths" of the quality function in the x'-direction and y'-direction, at points beyond which the amplitude of this function is negligible.

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we must, in the case of a spot-scan system, take — to eliminate confusion of the first kind — a number of independent samples N_1 given by

$$N_1 = 16a'b' \frac{1}{\Delta x} \frac{1}{\Delta y} , \tag{7-4}$$

so that the required time-bandwidth B is now given by

$$B = \frac{v}{\Delta x} . \tag{7-5}$$

The total number of independent samples contained in a waveform of bandwidth B and duration T is

$$N_1 = 2BT. \tag{7-6}$$

It follows, by comparing (7-4) and (7-6), that

$$2BT = 16a'b' \frac{1}{\Delta x} \frac{1}{\Delta y}$$

or, using (7-5),

$$2 \frac{v}{\Delta x} T = 16a'b' \frac{1}{\Delta x \Delta y} , \tag{7-8}$$

and therefore the length of the transmissivity-pattern displacements vT during one integration interval must at least be equal to

$$vT = 8a'b' \frac{1}{\Delta y} . \tag{7-9}$$

If the frequency-response of the transmissivity patterns has a maximum significant space-frequency

$$f_{y_{max}} = \frac{1}{\Delta y} , \tag{7-10}$$

the corresponding length of the transmissivity pattern must at least be equal to

$$vT = 8a'b' \frac{1}{\Delta y} \tag{7-11}$$

in order to avoid confusion of the first kind. The corresponding number of lines scanned is

$$J = 2\frac{f_{y_{max}}}{2b'} = \frac{vT}{2a'} . \tag{7-12}$$

This value of J provides perfect replication in a spot-scan system in the absence of "effective transmitter noise" or "effective receiver noise"[1]. The term effective transmitter noise is used to denote the noise originating in the low-level circuits of the transmitter which will probably consist of a photosensor and a pre-amplifier following this photosensor. The term "effective receiver noise" is used to denote the noise originating in the communications link and in the low-level circuits of the receiver.

Let now J_f be the number of scanning-lines chosen to get, with a spot-scan system, a replica of a given quality in the absence of effective photosensor and effective receiver noise. This number J_f will be smaller than that give by (7-12) if some confusion noise is acceptable. The corresponding transmissivity-pattern length vT is given by

$$vT = vT_f \equiv 2a'J_f \; . \qquad 7\text{-}13$$

We shall refer to the integration interval T_f, which corresponds to J_f, as the "frame time".

We shall assume in the discussion given in this section that J_f has been chosen so as to give, together with a given transmissivity-pattern element (which we do not specify), a replica of a specified quality in the y-direction.

The number of scanning lines J_f chosen not only determines the amount of confusion noise generated but also places an upper limit on the achievable resolution in the y-direction when all the points of the original image are scanned at least once by the aperture (that is, when no gaps exist between the image strips scanned by this aperture). The actual resolution achieved in this direction depends on the size, shape, and shading, of the aperture (transmissivity-pattern element) chosen; let the maximum space-frequency corresponding to this resolution be

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1 This is the case in practice if the original image intensity is high and if the level of the received signal is high.

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$$(f_y)_f \equiv (\frac{1}{\Delta y})_f \ . \tag{7-14}$$

On the other hand, the resolution in the x-direction depends solely on the size, shape, and shading of the aperture chosen; let the maximum space-frequency corresponding to this resolution be

$$(f_x)_f \equiv (\frac{1}{\Delta x})_f \ . \tag{7-15}$$

We then have for the corresponding time-bandwidth B_f, the relation

$$B_f = (\frac{v}{\Delta x})_f \ . \tag{7-16}$$

We now chose values for $(\Delta x)_f$ and $(\Delta y)_f$. These choices, together with our previous choice for J_f, determine uniquely the quality of the replica of a spot-scan system when the dimensions $2a'$ and $2b'$ of this replica are given. (We have assumed, see Assumption B , that the dimensions of the replica are the same as those of the original image.) From (7-13) and (7-15), we have

$$\frac{J_f}{(\Delta x)_f} = \frac{B_f T_f}{2a'} \ , \tag{7-17}$$

and hence the quality of the replica is also determined uniquely by the product $B_f T_f$ and $(\Delta y)_f$ for given values of $2a'$ and $2b'$.

We now seek to discover, in the case of a spot-scan system, the effect on the replica signal-to-noise ratio of the system parameters listed in Subsection 4.3. To simplify the discussion, we shall group together parameters that affect the replica signal-to-noise ratio in the same way, and shall use new quantities to describe the effect of each of these groups on this ratio. We shall use the total average power P_i intercepted by the receiver instead of parameters (5) and (6) in Subsection 4.3; we shall use N_r to denote the effective receiver noise per unit bandwidth, that is, the noise arising from items (4) and (5); and we shall use $(S/N)_t$ to denote the effective signal-to-noise ratio of the transmitted signal corresponding to item (3). The remaining two parameters listed in Subsection 4.3 are the video bandwidth B and the integrating time T. These two parameters are used explicitly in our present discussion.

The total power radiated by the original image through an aperture of the transmissivity pattern is at any given instant t' equal to $AI(x',y')$, where A is the effective area of this aperture and (x',y') its position at that instant. That is, the factor A is not necessarily equal to the geometric area of the aperture: this factor is adjusted to allow for a transmissivity not equal to unity and for a variation in the intensity $I(x',y')$ over the aperture.

Let D_e be the "effective detectivity" of the photosensor on which the radiation from the original image is focused, (after it goes through the aperture). (The term "effective" is used to indicate that the quantity D_e includes the effect of noise originating in the preamplifier following the photosensor.) Then the instantaneous signal-to-noise power ratio of the transmitted signal is

$$\frac{D_e^2 A^2 I^2(x',y')}{B} \quad . \tag{7-18}$$

If we transmit only the ac component of this signal, we have for the average transmitted signal-to-noise power ratio $(S/N)_t$ over the frame time T_f the expression

$$\left(\frac{S}{N}\right)_t = \frac{A^2 \sigma_I^2 D_e^2}{B} \quad , \tag{7-19}$$

where

$$\sigma_I^2 \equiv \overline{I^2}(x',y') - \bar{I}^2(x',y') \tag{7-20}$$

is the space-variance of $I(x',y')$ over the "frame" of the original image.

When the apertures are rectangular, this result is completely independent of the ratio of the y-dimension of the aperture over the distance between the center-lines of two adjacent image strips scanned by the aperture.[1] In effect, if this ratio is n, the number of times the same point (x',y') is scanned is also n; we therefore have

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[1] Except for a change in the "effective area" A, which is insignificant for the usual range of the values of A which one might consider using for a given value of J_f and $2b'$.

$$\sigma_I^2 = \overline{nI^2} - \overline{(nI)^2} = \overline{I^2} - \bar{I}^2 \quad . \tag{7-21}$$

This result is not precisely correct if the aperture is not rectangular or if adjacent strips are not contiguous; but it still holds, to a high degree of approximation, for the usual aperture shapes and centerline distances used.

The total average input signal power S_i to the receiver, which arises from the power radiated by the original image, is given by

$$S_i = P_i \frac{(\frac{S}{N})_t}{1+(\frac{S}{N})_t} \quad ; \tag{7-22}$$

and the total input noise power N_i to the receiver, which arises from the effective transmitter noise, is given by

$$N_i = P_i \frac{1}{1+(\frac{S}{N})_t} \quad . \tag{7-23}$$

If a point of the replica is scanned n times by the scanning aperture in the receiver (because the ratio of the y-dimension of the aperture over the distance between adjacent scanning lines is greater than unity), the space-variance σ_R^2 of the replica arising from the radiation from the original image is

$$\sigma_R^2 = n^2 S_i \tag{7-24}$$

because of the integrating property of the screen on which the replica is displayed and because the intensity of the original image is time-invariant during the integration interval. On the other hand, the space-variance σ_N^2 arising from both the input noise N_i and the receiver noise $N_r B$ is

$$\sigma_N^2 = nN_i + nN_r B \tag{7-25}$$

again because of the integrating property of the screen and because the noise fluctuates during the integration interval. Similarly, if the total integration interval T is larger than T_f, so that

$$T = mT_f, \tag{7-26}$$

where m is usually a (positive) integer, we have

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$$\sigma_R^2 = m^2 n^2 S_i \tag{7-27}$$

and

$$\sigma_N^2 = mnN_i + mnN_r B. \tag{7-28}$$

Consequently the replica signal-to-noise ratio $(S/N)_R$ is

$$\left(\frac{S}{N}\right)_R \equiv \frac{\sigma_R^2}{\sigma_N^2} = \frac{m^2 n^2 S_i}{mnN_i + mnN_r B} = \frac{mnS_i}{N_i + N_r B} \quad . \tag{7-29}$$

Replacing S_i and N_i by the expressionsgiven in (7-22) and (7-23), we obtain

$$\left(\frac{S}{N}\right)_R = \frac{\dfrac{mnP_i\left(\frac{S}{N}\right)_t}{1 + \left(\frac{S}{N}\right)_t}}{\left(\dfrac{P_i}{1 + \left(\frac{S}{N}\right)_t} + N_r B\right)} \qquad \text{7-30}$$

Now

$$N_r B = \frac{P_i}{\left(\frac{S}{N}\right)_r} \quad , \tag{7-31}$$

where $(S/N)_r$ is the effective receiver signal-to-noise corresponding to noise originating in the communications link and the receiver. Substituting this value of $N_r B$ into (7-30), we obtain

$$\left(\frac{S}{N}\right)_R = \frac{mn}{\dfrac{1}{\left(\frac{S}{N}\right)_t} + \dfrac{1}{\left(\frac{S}{N}\right)_r} + \dfrac{1}{\left(\frac{S}{N}\right)_t\left(\frac{S}{N}\right)_r}} \tag{7-32}$$

We now define C_f by

$$C_f \equiv B_f T_f \; , \tag{7-33}$$

or, using (7-26)

$$C_f = B \, \frac{T}{m} \quad . \tag{7-34}$$

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From equations (7-19), (7-31), and (7-34), we obtain

$$\left(\frac{S}{N}\right)_R = \frac{mn}{\dfrac{B}{A^2\sigma_I^2 D_e^2} + \dfrac{N_r B}{P_i} + \dfrac{B}{A^2\sigma_I^2 D_e^2}\ \dfrac{N_r B}{P_i}} \tag{7-35}$$

or

$$\left(\frac{S}{N}\right)_R = \frac{n\dfrac{T}{C_f}}{\dfrac{1}{A^2\sigma_I^2 D_e^2} + \dfrac{N_r}{P_i} + \dfrac{1}{A^2\sigma_I^2 D_e^2}\ \dfrac{N_r}{P_i}\ \dfrac{mC_f}{T}} \ . \tag{7-36}$$

Clearly the best replica signal-to-noise ratio occurs for

$$m = 1 . \tag{7-37}$$

That is, if we are given an integration interval T and desire a given replica quality in the absence of effective transmitter and receiver noise, we should — in order to get the best replica signal-to-noise ratio — chose to send a single frame during T. Consequently, in the case of a spot-scan system, we always chose

$$T_f = T . \tag{7-38}$$

Because, as mentioned earlier, the product

$$C_f = B_f T_f \tag{7-33}$$

is a constant for a given replica quality in the absence of effective transmitter and receiver noise, it follows that — under the stipulated conditions—we should use a bandwidth B given by

$$B = \frac{C_f}{T} \ , \tag{7-39}$$

so that

$$B = B_f . \tag{7-40}$$

With this choice of m, we have

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$$(\frac{S}{N})_R = \frac{n \frac{T}{c_f}}{\frac{1}{A^2 \sigma_I^2 D_e^2} + \frac{N_r}{P_i}\left\{1 + \frac{(C_f / T)}{A^2 \sigma_I^2 D_e^2}\right\}} \qquad (7\text{-}41)$$

We note, as expected, that the replica signal-to-noise ratio in both (7-36) and (7-41) increases with the integration interval T, and increases with decrease in c_f.

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7.2 Area-Scan System

We consider here only an area-scan system using quasi-random transmissivity patterns with internal periodicity because such patterns provide, in general, a superior performance to that provided by purely random patterns.

The former patterns have opaque regions of length 2a' between successive segments of length $2a'J_f$. Therefore, we now have

$$T = m\frac{J_f+1}{J_f}T_f \tag{7-42}$$

and

$$C_f = B_f T_f \tag{7-43}$$

instead of (7-33) and (7-34), respectively. When, as before, C_f and J_f are constants of the system and when T is given, we shall write

$$C_f = B\frac{T}{m}\frac{J_f}{J_f+1} \tag{7-44}$$

and consider the replica signal-to-noise ratio obtained for different values of B and m.

The analysis proceeds as before, except for the following differences.

First, the transmitted signal-to-noise power ratio $(S/N)_t'$ is now

$$\left(\frac{S}{N}\right)_t' = \frac{KA^2\sigma_I^2D_e^2}{B} \tag{7-45}$$

instead of as given in (7-19); the constant K is the number of apertures (transmissivity-pattern elements) illuminated at the same time. We now have

$$S_1 = P_i\frac{\left(\frac{S}{N}\right)_t'}{1+\left(\frac{S}{N}\right)_t'} \tag{7-46}$$

and

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$$N_i' = P_i \frac{(\frac{S}{N})_t'}{1 + (\frac{S}{N})_t'} \quad . \tag{7-47}$$

Second, the K apertures at the transmitter and the K apertures at the receiver give a total number K^2 of paths for the radiation arising from the original image. The desired replica is produced by radiation using K of these paths; the remaining (K^2-K) paths produce confusion noise of the second kind. We get

(a) for the spatial variance $\sigma_R'^2$ of the desired replica

$$\sigma_R'^2 = K m^2 n^2 S_i \ , \tag{7-48}$$

(b) for the space-variance $\sigma_{R2}'^2$ of the confusion noise

$$P_c \equiv \sigma_{R2}'^2 = (K-1) m n^2 S_i \tag{7-49}$$

(c) and for the space-variance of the replica noise, arising from both effective transmitter and receiver noises

$$\sigma_N'^2 = KnmN_i' + KnmN_r'B. \tag{7-50}$$

Consequently, the replica signal-to-noise ratio $(S/N)_R'$ is

$$(\frac{S}{N})_R' \equiv \frac{\sigma_R'^2}{\sigma_{R2}'^2 + \sigma_N'^2} = \frac{Km^2n^2S_i}{Kmn(N_i'+N_r'B) + P_c} \ . \tag{7-51}$$

Replacing S_i', N_i', and P_i, by the expressions given by (7-46), (7-47) and (7-49), we obtain

$$(\frac{S}{N})_R' = \frac{\dfrac{Km^2n^2P_i(\frac{S}{N})_t'}{1 + (\frac{S}{N})_t'}}{\dfrac{KmnP_i}{1 + (\frac{S}{N})_t'} + KmnN_r'B + \dfrac{(K-1)mn^2P_i(\frac{S}{N})_t'}{1 + (\frac{S}{N})_t'}} \tag{7-52}$$

or, remembering (7-31) and using (7-45),

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$$\left(\frac{S}{N}\right)'_R = \frac{mn}{\frac{1}{\left(\frac{S}{N}\right)'_t} + \frac{1}{\left(\frac{S}{N}\right)'_r} + \frac{1}{\left(\frac{S}{N}\right)'_t\left(\frac{S}{N}\right)'_r} + n\left(1-\frac{1}{K}\right)} \tag{7-53}$$

Substituting the expressions given by (7-45), (7-31) (with appropriate primes), and (7-39), in (7-53), we obtain

$$\left(\frac{S}{N}\right)'_R = \frac{nB\frac{T}{C_f}\frac{J_f}{J_f+1}}{\frac{B}{KA^2\sigma_I^2D_e^2} + \frac{N_rB}{P_i} + \frac{B}{KA^2\sigma_I^2D_e^2}\cdot\frac{N_rB}{P_i} + n\left(1-\frac{1}{K}\right)} \tag{7-54}$$

or

$$\left(\frac{S}{N}\right)'_R = \frac{n\frac{T}{C_f}\frac{J_f}{J_f+1}}{\frac{1}{KA^2\sigma_I^2D_e^2} + \frac{N_r}{P_i} + \frac{1}{KA^2\sigma_I^2D_e^2}\frac{N_r}{P_i}\frac{mC_f}{T}\frac{J_f+1}{J_f} + \frac{n}{C_f}\frac{T}{m}\frac{J_f}{J_f+1}\left(1-\frac{1}{K}\right)}\,; \tag{7-55}$$

hence, for large values of J_f,

$$\left(\frac{S}{N}\right)'_R \approx \frac{n\frac{T}{C_f}}{\frac{1}{KA^2\sigma_I^2D_e^2} + \frac{N_r}{P_i}\left(1+\frac{mC_f/T}{KA^2\sigma_I^2D_e^2}\right) + \frac{nT}{mC_f}\left(1-\frac{1}{K}\right)}\,. \tag{7-56}$$

We can, in principle, maximize $(S/N)'_R$ by choosing m so that

$$m^2 = (K-1)\frac{nP_iA^2\sigma_I^2D_e^2}{N_r\left(\frac{C_f}{T}\right)^2\left(1+\frac{1}{J_f}\right)^2} \tag{7-57}$$

or, for large values of J_f, so that

$$m^2 \approx (K-1)\frac{nP_iA^2\sigma_I^2D_e^2}{N_r\left(\frac{C_f}{T}\right)^2}\,. \tag{7-58}$$

For this value of m

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$$\left(\frac{S}{N}\right)_R' = \frac{n\frac{T}{C_f}\,\frac{J_f}{J_f+1}}{\left[\frac{1}{KA^2\sigma_I^2D_e^2} + \frac{N_r}{P_i} + 2\frac{(K-1)}{K}^{1/2}\left\{\frac{nN_r}{P_iA^2\sigma_I^2D_e^2}\right\}\right]^{1/2}} \cdot \qquad (7-59)$$

Hence, for large values of K and J, we have

$$\left(\frac{S}{N}\right)_R' \approx \frac{n\frac{T}{C_f}}{\frac{N_r}{P_i}} = \frac{nTP_i}{C_fN_r} = n\left(\frac{S}{N}\right)_r \qquad (7-60)$$

By comparing (7-59) with (7-36), we note that an area-scan system can, in principle, eliminate the effects of the effective transmitter noise completely. This is achieved with the same transmitter power but at the cost of additional bandwith. This bandwidth B is approximately m times greater than that required to produce a replica of a given quality in the absence of effective transmitter noise when the time T taken to produce a single replica is the same. The exact relation between the video-bandwidth of the spot-scan system and that of the area-scan system discussed in this section is

$$B_{\text{area-scan}} = m\frac{J_f + 1}{J_f} B_{\text{spot-scan}} \cdot \qquad (7-61)$$

This relation is obtained by comparing equation (7-39) with equation (7-44).

8. MERITS OF MECHANICALLY SCANNING APERTURES FOR SPOT-SCAN TELEVISION

Area-scan television uses simultaneously-illuminated multiple-scanning apertures. These cannot readily be instrumented with conventional pick-up tubes and cathode-ray tubes. We have, therefore, considered a variety of mechanically scanning apertures and, in particular, film strips, steel strips, and rigid discs. Systems using such scanning apertures are light and reliable and, more specifically, they need no high-voltage supplies and no sweep circuits.

We therefore believe that spot-scan television, using mechanically scanning apertures, should be considered for applications in which these features are valuable. This is particularly true when colored replicas are desired. Such replicas can, in effect, be obtained quite simply by using

(a) three scanning apertures that are transparent to three different wavelengths,
(b) three photosensitors to convert the radiant image into electrical signals,
(c) three separate channels in the communication link,
(d) three "driving" lamps at the receiver to provide the uniform illumination R_U, and
(e) a three-color integrating screen.

Note: Although we have not given in this report the mathematical analysis of multi-colored images, it can be shown that replicas of such images can also be obtained to any desired degree of perfection.

With film and steel strips, speeds in excess of 100 inches/sec are readily achievable, and with discs peripheral speeds in excess of 3000 inches/sec are readily achievable.

The application of interest to JPL — according to post-contractual discussions —is the transmission of an image 11 millimeters square. The desired resolution is equivalent to that of a 200-line (television) scan with the same resolution in a direction parallel to the scanning motion. The desired integration time T is 800 secs. For these parameters, it follows that the time available for scanning a single line is 4 secs, and the corresponding scanning-aperture speed is less than 3 mm/sec ($\approx$0.12 inch/sec). The accuracy with which this aperture must be placed is approximately 0.01 mm ($\approx$0.004 inch).

Under these conditions, we believe that a film strip — if the environmental conditions are not too severe — or a steel strip offers a suitable mechanization of the transmissivity patterns used for scanning. The length of the strip is 2.2 meters, which is quite acceptable.

We conclude by noting that a mechanically scanning aperture can be used at the transmitter together with a cathode-ray tube display at the receiver. This may provide the optimum instrumentation when high frame speeds and large replicas are required.

9. RELATIVE MERITS OF AREA-SCAN AND SPOT-SCAN TELEVISION

Area-scan television provides secure transmission: a replica of the original image cannot be reconstructed by intercepting the video signal transmitted if the exact form of the weighting functions used is not known.

In addition, for a given transmission time T per picture, an area-scan system can, in principle, produce — at the expense of bandwidth — a replica of a higher quality than that provided by a spot-scan system when the intensity of the original image is low. This higher quality is achieved by making the noise generated in the low-level circuits of the transmitter negligible.

Whether this improvement in quality can actually be realized in practice, depends on the system parameters. (The value of m in Subsection 6.2 that gives the maximum replica signal-to-noise ratio depends on these parameters, as can be seen from (7-56). It also depends on the transmission time (integration time) T permissible, which in turn depends on the speed at which the images of the moving objects to be televised are moving. In effect, practical considerations limit the speed v of the transmissivity patterns, and hence also the maximum displacement of these patterns during T. Furthermore, since — as we have seen in Section 5 — these transmissivity patterns must, in order to reduce "confusion noise" be much longer than those required to achieve a replica of a given quality in the case of a spot-scan system. Consequently, an area-scan system cannot televise moving objects at speeds as high as those which can be televised with a spot-scan system.

In deciding whether to use an area-scan system of the type we have discussed in this report, we must first determine whether the required transmissivity-pattern speed is achievable. If it is, one must then examine the system parameters to determine whether we are able to choose a near-optimum value for the parameter m.

Mechanically scanning apertures offer, also in the case of an area-scan system, a relatively light and reliable technique for providing colored replicas — especially insofar as the transmitter is concerned. In this case, we must add —in addition to the items listed on page 8.1 — three "quenching" lamps of different color at the receiver to subtract an appropriate amount of the dc component of the replica intensity distribution; we must also use an integrating screen which can be quenched with these lamps.